RESEARCH ARTICLE

On the equality of three formulas for Brumer–Stark units

Samit Dasgupta[1]　|　**Matthew H. L. Honnor[2]** 🆔　|　**Michael Spieß[3]**

[1]Department of Mathematics, Duke
University, Durham, North Carolina, USA

[2]St. Paul's School, London, UK

[3]Faculty of Mathematics, Bielefeld
University, Bielfeld, Germany

Correspondence
Matthew H. L. Honnor, St. Paul's School,
Lonsdale Road, London, SW13 9JT, UK.
Email: matthew24024@gmail.com

Abstract

We prove the equality of three conjectural formulas for
Brumer–Stark units. The first formula has essentially
been proven, so this paper also verifies the validity of the
other two formulas.

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Contents

1 | INTRODUCTION

In this paper, we prove the equality of three conjectural formulas for Brumer–Stark units made by the first and third authors in [4, 8], and [9].

One significance of this result is that the first formula has essentially been proven in [6], so this paper also verifies the validity of the other two formulas. Additionally, the third formula, made in [9], relates to a conjecture for the principal minors of the Gross–Regulator matrix. The validation of the third formula here gives a proof of this conjecture for the diagonal entries. Our work generalizes a partial result in this direction established in [14]. See [6] for a discussion of the application of the these formulas toward the explicit class field theory.

We now describe our results more precisely. Let F denote a totally real field, and let H denote a finite abelian extension of F. Write $G = \mathrm{Gal}(H/F)$. Let R_∞ denote the set of Archimedean places of F. Let R be a finite set of places of F containing R_∞ and the places that are ramified in H. Fix a prime ideal $\mathfrak{p} \notin R$ that splits completely in H and let $S = R \cup \{\mathfrak{p}\}$. Finally, we consider an auxiliary finite set T of primes of F, disjoint from S and satisfying a standard minor condition (see Section 2.2). From Section 3, and for our main results, we make the simplifying assumption that T consists of a single prime λ. The following conjecture was first stated by Tate and called the Brumer–Stark conjecture, [18, Conjecture 5.4].

Conjecture 1.1. *Let $\mathfrak{P}$ be a prime in H above $\mathfrak{p}$. There exists an element*

$$u_T \in \mathcal{U}_{\mathfrak{p}} = \{u \in H^* : \mid u \mid_v = 1 \text{ if } v \text{ does not divide } \mathfrak{p}\}$$

such that $u_T \equiv 1 \pmod{T}$, and for all $\sigma \in G$, we have $\mathrm{ord}_{\mathfrak{P}}(u_T^\sigma) = \zeta_{R,T}(H/F, \sigma, 0)$.

Here, v ranges over all finite and Archimedean places of H; in particular, each complex conjugation in H acts as an inversion on $\mathcal{U}_{\mathfrak{p}}$. The definition of the partial zeta function $\zeta_{R,T}(H/F, \sigma, 0)$ is recalled in Section 2.2. The conjectural element $u_T \in \mathcal{U}_{\mathfrak{p}}$ satisfying Conjecture 1.1 is called the Brumer–Stark unit for the data $(S, T, H, \mathfrak{P})$.

Conjecture 1.1 has been recently proved by the first author and collaborators (see [5, 7]). It is convenient for us to package together u_T and its conjugates over F into an element of $H^* \otimes \mathbb{Z}[G]$ that we call the Brumer–Stark element:

$$u_{\mathfrak{p}} = \sum_{\sigma \in G} u_T^\sigma \otimes [\sigma^{-1}] \in H^* \otimes \mathbb{Z}[G].$$

There have been three formulas conjectured for the image of the Brumer–Stark element $u_{\mathfrak{p}}$ in $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$. In [4], the first author conjectured a p-adic analytic formula for $u_{\mathfrak{p}}$ following the methods of Shintani and Cassou-Noguès. We denote this formula by u_1 and state it precisely in

Section 3. The other two formulas, which we denote as u_2 and u_3, were defined in joint work of the first and third authors in [8] and [9], respectively. Both of these formulas are cohomological in nature and are defined using the Eisenstein cocycle. They are stated precisely in Sections 5 and 6. We remark that our definition for $u_3 \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ differs from that used in [9] by a sign (which acts as inversion on the left factor of the tensor product or negation on the right side of the tensor product), in order to state our results more cleanly. The following combines the conjectures of the first author (for $i = 1$) and the first and third authors with (for $i = 2, 3$).

Conjecture 1.2. *For $i = 1, 2, 3$ we have $u_i = u_{\mathfrak{p}}$.*

The main result of this paper is the following.

Theorem 1.3. *We have the following equalities between the three conjectural formulas for the Brumer–Stark element $u_{\mathfrak{p}}$, namely,*

$$u_1 = u_2 = u_3 \text{ in } F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G].$$

Recent work of the first author with Kakde has proved that $u_1 = u_{\mathfrak{p}}$ up to a root of unity under some mild assumptions. Write $\mu(F_{\mathfrak{p}}^*)$ for the group of roots of unity in $F_{\mathfrak{p}}^*$.

Theorem 1.4 (Theorem 1.6, [6]). *Suppose that the rational prime p below $\mathfrak{p}$ is odd and unramified in F. Suppose further that there exists $\mathfrak{q} \in S$ that is unramified in H whose associated Frobenius $\sigma_{\mathfrak{q}}$ is the complex conjugation in G. Then, Conjecture 1.2 for u_1 holds up to multiplication by a root of unity in $F_{\mathfrak{p}}^*$:*

$$u_1 = u_{\mathfrak{p}} \text{ in } (F_{\mathfrak{p}}^*/\mu(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G].$$

Remark 1.5. Theorem 1.3 implies that $u_2 = u_3 = u_{\mathfrak{p}}$ in $(F_{\mathfrak{p}}^*/\mu(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G]$ under the assumptions of Theorem 1.4.

In Section 7, we prove that $u_2 = u_3$ via a direct cohomological calculation that was foretold in [9]. The proof that $u_1 = u_3$, which takes up Section 8, is more interesting and involves a new idea not present in prior work in this direction. It can be broken into two parts. Suppose that H/F is a CM abelian extension of conductor $\mathfrak{f}$ such that $\mathfrak{p}$ splits completely in H. We note that if $\mathfrak{q} \mid \mathfrak{f}$ then we must have $\mathfrak{q} \in R$. Denote by $E_+(\mathfrak{f}) \subset \mathcal{O}_F^*$ the subgroup of totally positive units congruent to 1 modulo $\mathfrak{f}$. We then prove by a direct calculation that

$$u_1(\sigma) \equiv u_3(\sigma) \pmod{E_+(\mathfrak{f})}, \tag{1}$$

where $u_i(\sigma)$ denotes the σ component of u_i.

Next, let $\mathfrak{f}'$ be an auxiliary ideal of $\mathcal{O}_F$ that is divisible only by primes dividing $\mathfrak{f}$. Let $H' \supset H$ be another finite abelian CM extension of F in which $\mathfrak{p}$ splits completely, such that the conductor of H'/F divides $\mathfrak{f}\mathfrak{f}'$. In particular, the extension H'/F is unramified outside R. For each $\sigma \in G$, we then show the norm compatibility relation for $i = 1, 3$,

$$u_i(\sigma, H) = \prod_{\substack{\tau \in \mathrm{Gal}(H'/F) \\ \tau|_H = \sigma}} u_i(\tau, H'). \tag{2}$$

We remark here that showing the above equation first requires the proof that $u_2 = u_3$. Applying (1) with H replaced by H' and combining with (2), we obtain

$$u_1(\sigma, H) \equiv u_3(\sigma, H) \pmod{E_+(\mathfrak{f}\mathfrak{f}')}. \tag{3}$$

If $R \neq R_\infty$, then taking larger and larger conductors $\mathfrak{f}\mathfrak{f}'$ and passing to a limit, we obtain the desired result

$$u_1(\sigma, H) = u_3(\sigma, H).$$

In the case $R = R_\infty$, we are required to do a little more work. The issue in this case is that $\mathfrak{f} = 1$ so there are no nontrivial ideals $\mathfrak{f}'$ we may take.

In this case, by adding auxiliary primes into R, we are able to show that there exists $\varepsilon \in E_+$ such that for each $\sigma \in G$ we have

$$u_1(\sigma, H) = \varepsilon u_3(\sigma, H).$$

We then extend the definitions for u_1 and u_2 to work with the trivial extension F/F. We note that this was already done for u_2 in [8]. In fact, u_2 is defined for any finite abelian extension H/F. Furthermore, in [8, Proposition 6.3] it is proved that $u_2(H/F) = 1$ if H has at least two real places. In particular, $u_2(F/F) = 1$. We also prove that $u_1(F/F) = 1$. By the norm compatibility property satisfied by u_1 and u_3 we have

$$1 = u_1(F) = \prod_{\sigma \in G} u_1(\sigma, H) = \varepsilon^{|G|} \prod_{\sigma \in G} u_3(\sigma, H) = \varepsilon^{|G|} \prod_{\sigma \in G} u_2(\sigma, H) = \varepsilon^{|G|}.$$

Thus, $\varepsilon = 1$. Therefore, $u_1 = u_3$.

2 | PRELIMINARIES FOR THE MULTIPLICATIVE INTEGRAL FORMULA

2.1 | Notation

Recall that we have let F be a totally real field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O} = \mathcal{O}_F$. Let $E_F = \mathcal{O}_F^*$ denote the group of global units. More generally, for a finite set S of nonarchimedean places of F we denote by $E_S = E_{F,S}$ the group of S-units of F. We define

$$\overline{S} = \{\mathfrak{q} : \mathfrak{q} \mid q \text{ where, for some } \mathfrak{r} \in S, \ \mathfrak{r} \mid q\}. \tag{4}$$

We also let H/F be a totally complex abelian extension containing a CM-subfield. Let $\mathfrak{f}$ denote the conductor of the extension H/F. We write $E_+(\mathfrak{f})$ for the totally positive units of F that are congruent to 1 $\pmod{\mathfrak{f}}$. Write $G_{\mathfrak{f}}$ for the narrow ray class group of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$ and suppose that $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. We write $\mathbb{O} = \mathcal{O}_{\mathfrak{p}} - \pi \mathcal{O}_{\mathfrak{p}} \subset F_{\mathfrak{p}}^*$.

Define $\mathbb{A} = \mathbb{A}_F$ as the adele ring of F. For a $\mathbb{Q}$-vector space W fix the notation $W_{\widehat{\mathbb{Z}}} = W \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = W \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{\infty}$. Here, $\mathbb{A}_{\mathbb{Q}}^{\infty}$ denotes the finite adeles of $\mathbb{Q}$. For an abelian group A and prime number ℓ, we put $A_{\ell} = A \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell}$.

For a place v of F we put $U_v = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ if $v \mid \infty$ and $U_v = \mathcal{O}_v^*$ if v is finite. For a set S of places of F we let $\mathbb{A}^S$ denote the adele ring away from S. We also define $U^S = \prod_{v \notin S} U_v$, and $U_S = \prod_{v \in S} U_v$. We shall also use the notations $F^S = (\mathbb{A}_F^S \times U_S) \cap F^*$ and $F_+^S = (\mathbb{A}_F^S \times U_S) \cap F_+^*$.

Furthermore, for a nonarchimedean place v of F and an integer $m \geqslant 0$ we let $U_v^{(m)}$ denote the mth higher units, that is, $U_v^{(m)} := \{x \in U_v \mid \mathrm{ord}_v(x - 1) \geqslant m\}$. If $\mathfrak{f}$ is an integral ideal and S is a finite set of places we then put

$$U_{\mathfrak{f}}^S := \prod_{v \notin S} U_v^{\mathrm{ord}_v(\mathfrak{f})}.$$

Finally, we note that if we have a function $f : X \to Z$, with $X \subseteq Y$ and Z an abelian group, then we can extend f to a function $f_! : Y \to Z$ by defining

$$f_!(y) = \begin{cases} f(y) & \text{if } y \in X \\ 0 & \text{if } y \in Y - X, \end{cases} \tag{5}$$

we call this the extension of f to Y by 0.

2.2 | Partial zeta functions

For $\sigma \in G = \mathrm{Gal}(H/F)$, we define the partial zeta function

$$\zeta_R(H/F, \sigma, s) = \sum_{\substack{(\mathfrak{a}, R) = 1 \\ \sigma_{\mathfrak{a}} = \sigma}} N\mathfrak{a}^{-s}. \tag{6}$$

Here, the sum ranges over all integral ideals $\mathfrak{a} \subset \mathcal{O}$ that are relatively prime to the elements of R and whose associated Frobenius element $\sigma_{\mathfrak{a}} \in G$ is equal to σ. The series (6) converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$, regular outside $s = 1$. When the field extension H/F is clear from context, we drop it from the notation and simply write $\zeta_R(\sigma, s)$. Since $\mathfrak{p}$ splits completely in H, the zeta functions associated with the sets of primes R and $S = R \cup \{\mathfrak{p}\}$ are related by the formula

$$\zeta_S(\sigma, s) = (1 - N\mathfrak{p}^{-s})\zeta_R(\sigma, s).$$

Recall that we have fixed an auxiliary finite set of primes of F, denoted as T, that is disjoint from S. The partial zeta function associated with the sets R and T is defined by the group ring equation

$$\sum_{\sigma \in G} \zeta_{R,T}(\sigma, s)[\sigma^{-1}] = \prod_{\eta \in T}(1 - [\sigma_{\eta}^{-1}]N\eta^{1-s}) \sum_{\sigma \in G} \zeta_R(\sigma, s)[\sigma^{-1}]. \tag{7}$$

We assume that the set T contains at least two primes of different residue characteristic or at least one prime η with absolute ramification degree at most $\ell - 2$, where η lies above ℓ. With this in place, the values $\zeta_{R,T}(K/F, \sigma, 0)$ are rational integers for any finite abelian extension K/F unramified outside R and any $\sigma \in \text{Gal}(K/F)$. This was shown by Deligne–Ribet [10] and Cassou–Noguès [1].

Our assumption on T implies that there are no nontrivial roots of unity in H that are congruent to 1 modulo T. Thus, the $\mathfrak{p}$-unit u_T in Conjecture 1.1, if it exists, is unique. Note also that our u_T is actually the inverse of the u in [13, Conjecture 7.4]. From Section 3 onwards, for ease of notation, we fix $T = \{\lambda\}$ for an appropriate choice of λ.

2.3 | Shintani zeta functions

Shintani zeta functions are a crucial ingredient in each of the constructions we study. We establish the necessary notation here, following Shintani [15]. In this subsection, we continue to allow any choice of appropriate T.

For each $v \in R_\infty$ we write $\sigma_v : F \to \mathbb{R}$ and fix the order of these embeddings. We can then embed F into $\mathbb{R}^n$ by $x \mapsto (\sigma_v(x))_{v \in R_\infty}$. Note that F^* acts on $\mathbb{R}^n$ with $x \in F^*$ acting by multiplication by $\sigma_v(x)$ on the v-component of any vector in $\mathbb{R}^n$. For linearly independent $v_1, \dots, v_r \in \mathbb{R}^n_+$, define the simplicial cone

$$C(v_1, \dots, v_r) = \left\{ \sum_{i=1}^r c_i v_i \in \mathbb{R}^n_+ : c_i > 0 \right\}.$$

Definition 2.1. A **Shintani cone** is a simplicial cone $C(v_1, \dots, v_r)$ generated by elements $v_i \in F \cap \mathbb{R}^n_+$. A **Shintani set** is a subset of $\mathbb{R}^n_+$ that can be written as a finite disjoint union of Shintani cones.

We now recall the definition of Shintani zeta functions. Let $\mathcal{O}_{F,\mathfrak{p}}$ denote the ring of $\mathfrak{p}$-integers of F. For any fractional ideal $\mathfrak{b} \subset F$ relatively prime to S, we let $\mathfrak{b}_\mathfrak{p} = \mathfrak{b} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$ denote the $\mathcal{O}_{F,\mathfrak{p}}$-module generated by $\mathfrak{b}$. Write $\mathfrak{f}$ for the conductor of the extension H/F. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to $S = \{\mathfrak{p}\} \cup R$ and $\overline{T}$, and let D be a Shintani set. For each compact open $U \subseteq F_\mathfrak{p}$, define, for $\text{Re}(s) > 1$,

$$\zeta_R(\mathfrak{b}, D, U, s) = \text{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in F \cap D, \, \alpha \in U \\ (\alpha, R) = 1, \, \alpha \in \mathfrak{b}_\mathfrak{p}^{-1} \\ \alpha \equiv 1 \pmod{\mathfrak{f}}}} \text{N}\alpha^{-s}.$$

For a general element $z \in F^*$ the congruence $z \equiv 1 \pmod{\mathfrak{f}}$ means that $z - 1 \in \mathfrak{f}\mathcal{O}_\mathfrak{f} \cap F$, where $\mathcal{O}_\mathfrak{f}$ is the $\mathfrak{f}$-adic completion of $\mathcal{O}_F$. We define $\zeta_{R,T}(\mathfrak{b}, D, U, s)$ in analogy with (7). Suppose that

$$\prod_{\eta \in T} (1 - [\eta]\text{N}\eta^{1-s}) = \sum_{\mathfrak{a}} c_\mathfrak{a}(s)[\mathfrak{a}]$$

in the group ring of fractional ideals with coefficients in the ring of complex valued functions on $\mathbb{C}$, and define

$$\zeta_{R,T}(\mathfrak{b}, D, U, s) = \sum_{\mathfrak{a}} c_\mathfrak{a}(s)\zeta_R(\mathfrak{a}^{-1}\mathfrak{b}, D, U, s). \tag{8}$$

In particular, if $N\eta = \ell$ and $T = \{\eta\}$, we have

$$\zeta_{R,T}(\mathfrak{b}, D, U, s) = \zeta_R(\mathfrak{b}, D, U, s) - \ell^{1-s}\zeta_R(\mathfrak{b}\eta^{-1}, D, U, s).$$

It follows from Shintani's work in [15] that the function $\zeta_{R,T}(\mathfrak{b}, D, U, s)$ has a meromorphic continuation to $\mathbb{C}$. We now want to define conditions on the set of primes T and the Shintani set D to allow our Shintani zeta functions to be integral at 0.

Definition 2.2. A prime ideal η of F is called **good** for a Shintani cone C if

- $N\eta$ is a rational prime ℓ;
- the cone C may be written $C = C(v_1, \dots, v_r)$ with $v_i \in \mathcal{O}$ and $v_i \notin \eta$.

We also say that η is **good** for a Shintani set D if D can be written as a finite disjoint union of Shintani cones for which η is good.

Definition 2.3. The set T is **good** for a Shintani set D if D can be written as a finite disjoint union of Shintani cones $D = \bigsqcup C_i$ so that for each cone C_i, there are at least two primes in T that are good for C_i (necessarily of different residue characteristic by our earlier assumption) or one prime $\eta \in T$ that is good for C_i such that $N\eta \geq n + 2$.

Remark 2.4. Given any Shintani set D, it is possible to choose a set of primes T such that T is good for D. In fact, all but a finite number of prime ideals with prime norm are good for a given Shintani set.

We can now note the required property to allow our Shintani zeta functions to be integral at zero. The proposition below is proved in [4, p. 15].

Proposition 2.5. *If the set of primes T is good for a Shintani set D, then*

$$\zeta_{R,T}(\mathfrak{b}, D, U, 0) \in \mathbb{Z}.$$

We define a $\mathbb{Z}$-valued measure $\nu_T(\mathfrak{b}, D)$ on $\mathcal{O}_\mathfrak{p}$ by

$$\nu_T(\mathfrak{b}, D, U) := \zeta_{R,T}(\mathfrak{b}, D, U, 0), \tag{9}$$

for $U \subseteq \mathcal{O}_\mathfrak{p}$ compact open.

We are mostly interested in a particular type of Shintani set, one which is a fundamental domain for the action of a finite index subgroup $V \subset E_+(\mathfrak{f})$.

Definition 2.6. Let $V \subset E_+(\mathfrak{f})$ be a finite index subgroup (which is necessarily free of rank $n - 1$). We call a Shintani set D a Shintani domain for V if D is a fundamental domain for the action of V on $\mathbb{R}_+^n$. That is,

$$\mathbb{R}_+^n = \bigsqcup_{\epsilon \in V} \epsilon D \quad \text{(disjoint union)}.$$

The existence of such domains follows the work of Shintani, in particular from [15, Proposition 4]. We note here some simple equalities that follow from the definitions. More details are given in [4, §3.3]. Recall we have written $G_{\mathfrak{f}}$ for the narrow ray class group of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$, and write $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. We denote by $H_{\mathfrak{f}}$ the narrow ray class field of F of conductor $\mathfrak{f}$, and by H the maximal subfield of $H_{\mathfrak{f}}$ containing F in which the prime $\mathfrak{p}$ splits completely. Let D be a Shintani domain for $E_+(\mathfrak{f})$ and write $\mathbb{O} = \mathcal{O}_{\mathfrak{p}} - \pi \mathcal{O}_{\mathfrak{p}}$. Then,

$$\nu_T(\mathfrak{b}, D, \mathbb{O}) = \zeta_{S,T}(H/F, \mathfrak{b}, 0) = 0, \quad \text{and} \quad \nu_T(\mathfrak{b}, D, \mathcal{O}_{\mathfrak{p}}) = \zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0).$$

We now give two technical definitions that are necessary in the definition of u_1.

Proposition 2.7. *Let $V \subset E_+(\mathfrak{f})$ be a finite index subgroup. Let D and D' be Shintani domains for V. We may write D and D' as finite disjoint unions of the same number of simplicial cones*

$$D = \bigcup_{i=1}^{d} C_i, \quad D' = \bigcup_{i=1}^{d} C_i', \tag{10}$$

with $C_i' = \epsilon_i C_i$ for some $\epsilon_i \in V$, $i = 1, \dots, d$.

Proof. [4, Proposition 3.15] proves this result when $V = E_+(\mathfrak{f})$. The proof of this proposition is analogous. $\square$

A decomposition as in (10) is called a simultaneous decomposition of the Shintani domains (D, D').

Definition 2.8. Let (D, D') be a pair of Shintani domains. A set T is **good** for the pair (D, D') if there is a simultaneous decomposition as in (10) such that for each cone C_i, there are at least two primes in T that are good for C_i, or there is one prime $\eta \in T$ that is good for C_i such that $N\eta \geqslant n + 2$.

Definition 2.9. Let D be a Shintani domain. If $\beta \in F^*$ is totally positive, then T is β-**good** for D if T is good for the pair $(D, \beta^{-1}D)$.

Lemma 2.10 (Lemma 3.20, [4]). *Let D be a Shintani set and U a compact open subset of $\mathcal{O}_{\mathfrak{p}}$. Let $\mathfrak{b}$ be a fractional ideal of F, and let $\beta \in F^*$ be totally positive so that $\beta \equiv 1 \pmod{\mathfrak{f}}$ and $\mathrm{ord}_{\mathfrak{p}}(\beta) \geqslant 0$. Suppose that $\mathfrak{b}$ and β are relatively prime to R and that $\mathfrak{b}$ is also relatively prime to $\overline{T}$. Let $\mathfrak{q} = (\beta)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\beta)}$. Then*

$$\zeta_{R,T}(\mathfrak{b}\mathfrak{q}, D, U, 0) = \zeta_{R,T}(\mathfrak{b}, \beta D, \beta U, 0).$$

We end this section with a lemma of Colmez that allows us to give an explicit Shintani domain. Let α be, up to a sign, one of the standard basis vectors of $\mathbb{R}^n$. Note that its ray $(\alpha \mathbb{R}_+)$ is preserved by the action of $\mathbb{R}_+^n$. We define $\overline{C}_\alpha(v_1, \dots, v_r)$ to be the union of the cone $C(v_1, \dots, v_r)$ with the boundary cones that are brought into the interior of the cone by a small perturbation by α, that

is, the set whose characteristic function is given by

$$\mathbb{1}_{\overline{C}_\alpha(v_1,\dots,v_r)}(x) = \lim_{h\to 0^+} \mathbb{1}_{C(v_1,\dots,v_r)}(x + h\alpha). \tag{11}$$

We use the usual bar notation for homogeneous chains

$$[x_1 \mid \cdots \mid x_{n-1}] = (1, x_1, x_1 x_2, \dots, x_1 \dots x_{n-1}).$$

Let $x_1, \dots, x_{n-1} \in F$. We define the sign map $\delta : F^n \to \{-1, 0, 1\}$ by the rule

$$\delta(x_1, \dots, x_n) = \text{sign}(\det(\omega(x_1, \dots, x_n))), \tag{12}$$

where $\omega(x_1, \dots, x_n)$ denotes the $n \times n$ matrix whose columns are the images of the x_i in $\mathbb{R}^n$. We adopt the convention $\text{sign}(0) = 0$.

Lemma 2.11 (Lemma 2.2, [3]). *Let α be, up to a sign, one of the standard basis vectors of $\mathbb{R}^n$. Let $\varepsilon_1, \dots, \varepsilon_{n-1} \in E_+(\mathfrak{f})$ such that $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle \subset E_+(\mathfrak{f})$ has finite index. Suppose that for all $\tau \in S_{n-1}$ we have*

$$\delta([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]) = \text{sign}(\tau).$$

Then, the Shintani set

$$D = \bigcup_{\tau \in S_{n-1}} \overline{C}_\alpha([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]),$$

is a Shintani domain for V.

For more details on the above lemma, we refer to [2, §1.3]. We note also that another proof of the lemma is given in [11, Corollary 2].

The existence of Shintani domains follows from the work of Shintani in [15]. In Lemma 4.3, we show the existence of units $\varepsilon_1, \dots, \varepsilon_{n-1} \in E_+(\mathfrak{f})$ that satisfy the conditions of Lemma 2.11.

3 | THE MULTIPLICATIVE INTEGRAL FORMULA (u_1)

Definition 3.1. Let I be an abelian topological group that may be written as an inverse limit of discrete groups

$$I = \varprojlim I_\alpha.$$

Denote the group operation on I multiplicatively. For each $i \in I_\alpha$, denote by U_i the open subset of I consisting of the elements that map to i in I_α. Suppose that G is a compact open subset of a quotient of $\mathbb{A}_F^*$. Let $f : G \to I$ be a continuous map, and let μ be a $\mathbb{Z}$-valued measure on G. We define the multiplicative integral, written with a cross through the integration sign, by

$$\fint_G f(x) d\mu(x) = \varprojlim \prod_{i \in I_\alpha} i^{\mu(f^{-1}(U_i))} \in I.$$

Let λ be a prime of F such that $\mathrm{N}\lambda = \ell$ for a prime number $\ell \in \mathbb{Z}$ and $\ell \geqslant n + 2$. We assume that no primes in S have residue characteristic equal to ℓ. In this section and from this point on, we take $T = \{\lambda\}$.

Definition 3.2. Let $\mathcal{D}$ be a Shintani domain for $E_+(\mathfrak{f})$, and assume that λ is π-good for $\mathcal{D}$. Define the error term

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi) := \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_\lambda(\mathfrak{b}, \epsilon \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} \in E_+(\mathfrak{f}). \tag{13}$$

By [4, Lemma 3.14], only finitely many of the exponents in (13) are nonzero. [4, Proposition 3.12] and the assumption that λ is π-good for $\mathcal{D}$ implies that the exponents are integers. We recall from (9) that the measure is defined as

$$\nu_\lambda(\mathfrak{b}, \epsilon \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) = \zeta_{R,\lambda}(\mathfrak{b}, \epsilon \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}, 0).$$

We are now ready to write down the conjectural formula from [4]. We note that for any Shintani domain $\mathcal{D}$ we can always choose a prime λ that is π-good for $\mathcal{D}$. In fact, all but a finite number of primes will satisfy this property. Henceforth, we can assume that λ satisfies the property written above and is π-good for $\mathcal{D}$. We now give the main definition of this section.

Definition 3.3. Let $\mathcal{D}$ be a Shintani domain for $E_+(\mathfrak{f})$, and assume that λ is π-good for $\mathcal{D}$. Define

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}) := \epsilon(\mathfrak{b}, \mathcal{D}, \pi) \pi^{\zeta_{R,\lambda}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0)} \fint_{\mathbb{O}} x \, d\nu_\lambda(\mathfrak{b}, \mathcal{D}, x) \in F_{\mathfrak{p}}^*.$$

As our notation suggests, we have the following proposition.

Proposition 3.4 Proposition 3.19, [4]. *The element $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D})$ does not depend on the choice of generator π of $\mathfrak{p}^e$.*

The following is conjectured.

Conjecture 3.5 (Conjecture 3.21, [4]). *Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$, and suppose that $\mathfrak{p}^e = (\pi)$ with π totally positive and $\pi \equiv 1 \pmod{\mathfrak{f}}$. Let $\mathcal{D}$ be a Shintani domain for $E_+(\mathfrak{f})$, and let λ be π-good for $\mathcal{D}$. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and ℓ. We have the following.*

1. *The element $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}) \in F_{\mathfrak{p}}^*$ depends only on the class of $\mathfrak{b} \in G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle$ and no other choices, including the choice of $\mathcal{D}$, and hence may be denoted as $u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}})$, where $\sigma_{\mathfrak{b}} \in \mathrm{Gal}(H/F)$.*
2. *The element $u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}})$ lies in $\mathcal{U}_{\mathfrak{p}}$, and $u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}}) \equiv 1 \pmod{\lambda}$.*
3. *Shimura reciprocity law: for any fractional ideal $\mathfrak{a}$ of F prime to S and to ℓ, we have*

$$u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{a}\mathfrak{b}}) = u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}})^{\sigma_{\mathfrak{a}}}.$$

As we noted in the introduction, this conjecture has been proved up to a root of unity (Theorem 1.4). We want to state the formula over $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ to match with the cohomological constructions.

Definition 3.6. We define

$$u_1 = \sum_{\mathfrak{b} \in G_\mathfrak{f}/\langle \mathfrak{p} \rangle} u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}) \otimes [\sigma_\mathfrak{b}^{-1}] \in F_\mathfrak{p}^* \otimes \mathbb{Z}[G].$$

3.1 | Transferring to a subgroup

In this section, we recall the results [14], which allow us to transfer to a subgroup. Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Recall that π is totally positive, congruent to 1 modulo $\mathfrak{f}$ and satisfies $(\pi) = \mathfrak{p}^e$, where e is the order of $\mathfrak{p}$ in $G_\mathfrak{f}$. Let $\mathcal{D}'_V$ be a Shintani set which is a fundamental domain for the action of V on $\mathbb{R}_+^n$ and assume that λ is π-good for $\mathcal{D}'_V$. As before, we shall refer to such Shintani sets as Shintani domains for V. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and ℓ.

We define

$$u_1(V, \sigma_\mathfrak{b}) = u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}'_V) := \prod_{\epsilon \in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b}, \epsilon \mathcal{D}'_V \cap \pi^{-1} \mathcal{D}'_V, \mathcal{O}_\mathfrak{p}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{D}'_V, \mathcal{O}_\mathfrak{p}, 0)} \fint_{\mathbb{O}} x \, d\nu_\lambda(\mathfrak{b}, \mathcal{D}'_V, x),$$

and write $u_1(V) = \sum_{\sigma \in G} u_1(V, \sigma) \otimes [\sigma^{-1}]$.

Proposition 3.7 (Proposition 6.11, [14]). *Let $\mathcal{K}$ and $\mathcal{K}'$ be two Shintani domains for V and λ a prime of F such that λ is π-good for $\mathcal{K}$ and $\mathcal{K}'$. If λ is also good for $(\mathcal{K}, \mathcal{K}')$, then $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{K}) = u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{K}')$.*

Let $V \subset E_+(\mathfrak{f})$ be a finite index subgroup. The following proposition shows the relation between $u_1(\sigma)$ and $u_1(V, \sigma)$.

Proposition 3.8 (Proposition 6.12, [14]). *Let $\mathcal{D}$ be a Shintani domain for $E_+(\mathfrak{f})$. Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Write $g_1, \dots, g_{n-1}$ for a $\mathbb{Z}$-basis of $E_+(\mathfrak{f})$ such that $g_1^{b_1}, \dots, g_{n-1}^{b_{n-1}}$ is a $\mathbb{Z}$-basis for V. Define*

$$\mathcal{D}_V := \bigcup_{j_1=0}^{b_1-1} \cdots \bigcup_{j_{n-1}=0}^{b_{n-1}-1} g_1^{j_1} \dots g_{n-1}^{j_{n-1}} \mathcal{D}.$$

Then, if $b_1, \dots, b_{n-1} > M$, where $M = M(\pi, g_1, \dots, g_{n-1})$ is some constant that depends on $g_1, \dots, g_{n-1}$ and π (up to multiplication by an element of $E_+(\mathfrak{f})$), we have

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}_V) = u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D})^{[E_+(\mathfrak{f}):V]}.$$

4 | PRELIMINARIES FOR THE COHOMOLOGICAL FORMULAS

4.1 | Continuous maps

For topological spaces X and Y, let $C(X, Y)$ denote the set of continuous maps $X \to Y$. If R is a topological ring we let $C_c(X, R)$ denote the subset of $C(X, R)$ of continuous maps with compact

support. If we consider Y (resp. R) with the discrete topology then we shall also write $C^0(X, Y)$ (resp. $C_c^0(X, R)$) instead of $C(X, Y)$ (resp. $C_c(X, R)$).

Assume now that X is a totally disconnected topological Hausdorff space and A a locally profinite group. We define subgroups $C^\diamond(X, A) \subseteq C(X, A)$ and $C_c^\diamond(X, A) \subseteq C_c(X, A)$ by

$$C^\diamond(X, A) = C^0(X, A) + \sum_K C(X, K),$$

$$C_c^\diamond(X, A) = C_c^0(X, A) + \sum_K C_c(X, K),$$

where the sums are taken over all compact open subgroups K of A. So, $C_c^\diamond(X, A)$ is the subgroup of $C_c(X, A)$ generated by locally constant maps with compact support $X \to A$ and by continuous maps with compact support $X \to K \subseteq A$ for some compact open subgroup $K \subseteq A$. Similarly, $C^\diamond(X, A)$ is the subgroup of $C(X, A)$ generated by locally constant maps $X \to A$ and by continuous maps $X \to K \subseteq A$ for some compact open K.

The following notation is used in the formulation of u_2. Given two arbitary finite, disjoint sets Σ_1, Σ_2 of places of F and a locally profinite group A we put

$$C_?(\Sigma_1, A)^{\Sigma_2} = C_?((\mathbb{A}_F^{\Sigma_2})^* / U^{\Sigma_1 \cup \Sigma_2}, A).$$

where $? \in \{\diamond, c, 0\}$. Here, for a set of places S, U^S denotes the subgroup of $\mathbb{A}_F^*$ of ideles $(x_v)_v$ with local components $x_v = 1$ if $v \in S$, $x_v > 0$ if $v \mid \infty$ and x_v is a local unit if $v \notin S \cup R_\infty$.

We also introduce a generalization of the above notation. For S_1, S_2 disjoint sets of places of F, let

$$C_?(S_1, S_2, A) = C_?\left(\prod_{\mathfrak{p} \in S_1} F_\mathfrak{p} \times (\mathbb{A}_F^{S_1})^* / U^{S_1 \cup S_2}, A\right).$$

If S_3 is an additional disjoint set of places we also define

$$C_?(S_1, S_2, A)^{S_3} = C_?\left(\prod_{\mathfrak{p} \in S_1} F_\mathfrak{p} \times (\mathbb{A}_F^{S_1 \cup S_3})^* / U^{S_1 \cup S_2 \cup S_3}, A\right).$$

4.2 | Measures

We now wish to attach to a homomorphism $\mu : C_c(X, \mathbb{Z}) \to \mathbb{Z}[G]$ an $A \otimes \mathbb{Z}[G]$-valued measure on X for any abelian group A and finite abelian group G. We write the group operation of A multiplicatively. First, we note that μ can be uniquely extended to a homomorphism of $\mathbb{Z}[G]$-modules $\mu : C_c(X, \mathbb{Z}[G]) \cong C_c(X, \mathbb{Z}) \otimes \mathbb{Z}[G] \to \mathbb{Z}[G]$. By tensoring μ with the identity map on A, we obtain a homomorphism of $\mathbb{Z}[G]$-modules

$$\mu_A : C_c(X, \mathbb{Z}) \otimes (A \otimes \mathbb{Z}[G]) \cong C_c^0(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]. \tag{14}$$

To write this map explicitly, we first note that the isomorphism in (14) is given by

$$f \otimes \alpha \mapsto \alpha \cdot f, \text{ with inverse } g \mapsto \sum_{\alpha \in A \otimes \mathbb{Z}[G]} (\alpha \otimes g_\alpha),$$

where $g_\alpha(x) = 1$ if $g(x) = \alpha$ and 0 otherwise. Here, we have $f \in C_c(X, \mathbb{Z})$, $\alpha \in A \otimes \mathbb{Z}[G]$ and $g \in C_c^0(X, A \otimes \mathbb{Z}[G])$. Thus, the homomorphism μ_A is given by

$$\mu_A(g) = \sum_{\alpha \in A \otimes \mathbb{Z}[G]} \left(\sum_{\sigma \in G} \sum_{\tau \in G} \alpha_\tau^{\mu_\sigma(g_\alpha)} \otimes \sigma\tau \right),$$

where $\alpha = \sum_{\tau \in G} \alpha_\tau \otimes \tau$, $\mu(g_\alpha) = \sum_{\sigma \in G} \mu_\sigma(g_\alpha)[\sigma]$ and g_α is as defined before. If A is profinite we can consider the homomorphism

$$\mu_A := \varprojlim_K \mu_{A/K} : \varprojlim_K C_c(X, A/K \otimes \mathbb{Z}[G]) \to \varprojlim_K A/K \otimes \mathbb{Z}[G] = A \otimes \mathbb{Z}[G]$$

where K ranges over the open subgroups of A. Since $C_c(X, A \otimes \mathbb{Z}[G]) \subseteq \varprojlim_K C_c(X, A/K \otimes \mathbb{Z}[G])$, we see that μ_A extends canonically to a homomorphism $C_c(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]$ (which we denote by μ_A as well). For a general A (not necessarily profinite), we have seen that μ induces a homomorphism $C_c(X, K \otimes \mathbb{Z}[G]) \to K \otimes \mathbb{Z}[G]$ for every compact open subgroup $K \subset A$. Combining these maps we see that μ induces a canonical homomorphism $\mu_A : C_c^\diamond(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]$. Define the set of $A \otimes \mathbb{Z}[G]$-valued measures on X to be

$$\mathrm{Meas}(X, A \otimes \mathbb{Z}[G]) = \mathrm{Hom}(C_c^\diamond(X, A \otimes \mathbb{Z}[G]), A \otimes \mathbb{Z}[G]).$$

The map $\mu \mapsto \mu_A$ defines a homomorphism $\mathrm{Hom}(C_c(X, \mathbb{Z}[G]), \mathbb{Z}) \to \mathrm{Meas}(X, A \otimes \mathbb{Z}[G])$.

In practice, we apply certain specialisations of the general construction above. In the definition of u_2, we construct $\mu \in \mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z})$ rather than in $\mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}[G])$. We include $\mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z})$ into $\mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}[G])$ by the map

$$\iota_1 : \mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(C_c(X, \mathbb{Z}), \mathbb{Z}[G]), \quad \iota_1(\mu)(f) = \mu(f)[\mathrm{id}],$$

for $f \in C_c(X, \mathbb{Z})$.

In the definition of u_3, we have a measure on A rather than on $A \otimes \mathbb{Z}[G]$. We include $C_c^\diamond(X, A)$ into $C_c^\diamond(X, A \otimes \mathbb{Z}[G])$ via the map

$$\iota_2 : C_c^\diamond(X, A) \to C_c^\diamond(X, A \otimes \mathbb{Z}[G]), \quad \iota_2(f)(x) = f(x) \otimes \mathrm{id}_G,$$

for $x \in X$.

4.3 | Eisenstein cocycles

We now define the Eisenstein cocycle. The cohomological constructions u_2 and u_3 require different variations.

Let $E_+(\mathfrak{f})_\mathfrak{p}$ denote the group of totally positive $\mathfrak{p}$-units of F that are congruent to 1 (mod $\mathfrak{f}$). The abelian group $E_+(\mathfrak{f})_\mathfrak{p}$ is free of rank n. For $x_1, \dots, x_n \in E_+(\mathfrak{f})_\mathfrak{p}$, a fractional ideal $\mathfrak{b}$ coprime to S and ℓ, and compact open $U \subset F_\mathfrak{p}$, we put

$$v_{\mathfrak{b},\lambda}^\mathfrak{p}(x_1, \dots, x_n)(U) = \delta(x_1, \dots, x_n)\zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}(x_1, \dots, x_n), U, 0).$$

Here, the Shintani zeta function is defined in (8), δ is defined in (12) and $\overline{C}_{e_1}(x_1, \dots, x_n)$ is defined in (11). Then, $v_{\mathfrak{b},\lambda}^\mathfrak{p}$ is a homogeneous $(n-1)$-cocycle on $E_+(\mathfrak{f})_\mathfrak{p}$ with values in the space of

$\mathbb{Z}$-distributions on $F_{\mathfrak{p}}$. This follows from [2, Theorem 2.6]. We obtain a class

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} := [\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}] \in H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \operatorname{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z})).$$

Remark 4.1. The function $\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}(x_1, \dots, x_n)$ is viewed as an element of $\operatorname{Hom}(C_c(F_{S_p}, \mathbb{Z}), \mathbb{Z}[G])$ via the following canonical integration pairing:

$$(f, \mu) \mapsto \int_{F_R} f(t)d\mu(t) = \lim_{\|\mathcal{V}\| \to 0} \sum_{V \in \mathcal{V}} f(t_V)\mu(V)$$

where the limit is over increasingly finer covers $\mathcal{V}$ of the support of f by compact open subgroups $V \subseteq F_{S_p}$ and $t_V \in V$ is any element of V.

We also define

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda} = \sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle} \operatorname{rec}_{H/F}(\mathfrak{b})\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \in H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \operatorname{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z}[G])), \tag{15}$$

where the sum ranges over a system of representatives of $G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle$. This construction is adapted from the construction of $\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda}$ in [9, §3.3].

We write W for F considered as a $\mathbb{Q}$-vector space, and $W_\infty = W \otimes_{\mathbb{Q}} \mathbb{R}$. As before, let λ be a prime of F such that $N\lambda = \ell$ for a prime number $\ell \in \mathbb{Z}$ and $\ell \geq n + 2$. We assume that no primes in S have residue characteristic equal to ℓ. Let $W_\ell = W \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Define $\phi_\lambda \in C_c(W_\ell, \mathbb{Z})$ by $\phi_\lambda = \mathbb{1}_{\mathcal{O}_F \otimes \mathbb{Z}_\ell} - \ell \mathbb{1}_{\lambda \otimes \mathbb{Z}_\ell}$, that is,

$$\phi_\lambda(v) = \begin{cases} 1 & \text{if } v \in (\mathcal{O}_F \otimes \mathbb{Z}_\ell) - (\lambda \otimes \mathbb{Z}_\ell), \\ 1 - \ell & \text{if } v \in \lambda \otimes \mathbb{Z}_\ell, \\ 0 & \text{if } v \in V_\ell - (\mathcal{O}_F \otimes \mathbb{Z}_\ell). \end{cases} \tag{16}$$

By fixing an ordering of the infinite places $v \in R_\infty$, we fix an identification $W_\infty \cong \mathbb{R}^n$. We define F^ℓ_+ as in Section 2.1. If D is a Shintani set and $\Phi \in C_c(W_{\widehat{\mathbb{Z}}}, \mathbb{Z})$ then, following [8], we define the Dirichlet series

$$L(D, \Phi; s) = \sum_{v \in W \cap D} \Phi(v)N(v)^{-s}. \tag{17}$$

It is known to converge for $\operatorname{Re}(s) > 1$ and extend to the whole complex plane except for possibly a simple pole at $s = 0$. Moreover, if D and Φ are as given in the following proposition then $L(D, \Phi; s)$ is holomorphic. We remark that the set S does not appear in the definition of this Dirichlet series. In the following proposition, we will decorate the L-function with λ since the choice of Φ incorporates λ into it.

Proposition 4.2. *Let $\omega_1, \dots, \omega_n \in F^\ell_+$. For a map $\phi \in C_c(W_{\widehat{\mathbb{Z}}^\ell}, \mathbb{Z})$, let*

$$\operatorname{Eis}^0_{F,\lambda}(\omega_1, \dots, \omega_n)(\phi) = \delta(\omega_1, \dots, \omega_n)L_\lambda(\overline{C}_{e_1}(\omega_1, \dots, \omega_n), \Phi; 0),$$

where $\Phi = \phi \otimes \phi_\lambda$. Then, $\operatorname{Eis}^0_{F,\lambda}$ is an F^ℓ_+-homogeneous $(n-1)$-cocycle yielding a class

$$\operatorname{Eis}^0_{F,\lambda} \in H^{n-1}(F^\ell_+, \operatorname{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\ell}, \mathbb{Z}), \mathbb{Z})).$$

Proof. This proposition follows the combination of [8, Definition 4.5] and [8, Lemma 5.1]. $\qquad\square$

We note that in [8] a more general cocycle $\mathrm{Eis}^0_{F,\lambda,v}$ is constructed. Here, $v \in R_\infty$ is a fixed infinite place. We explain the context of this now. For a subgroup $H \subseteq F^{\ell,v}$ and an H-module M, define $M(\delta) = M \otimes \mathbb{Z}(\delta)$. Thus, $M(\delta)$ is the group M with H-action given by $x \cdot m = \delta(x)xm$ for $x \in H$ and $m \in M$. Then, $\mathrm{Eis}^0_{F,\lambda,v} \in H^{n-1}(F^{\ell,v}, \mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\ell}, \mathbb{Z}), \mathbb{Z})(\delta))$ and we have the equality

$$\mathrm{res}^{F^{v,\ell}}_{F^\ell_+}(\mathrm{Eis}^0_{F,\lambda,v}) = \mathrm{Eis}^0_{F,\lambda}.$$

4.4 │ Colmez subgroups

In the definitions for the Eisenstein cocycle and its variants the sign map δ appears. For the explicit calculations, we want to perform later it is convenient if we can work with a finite index subgroup $V \subseteq E_+(\mathfrak{f})$ such that $V = \langle g_1, \dots, g_{n-1}\rangle$ and that we are able to choose π such that, after writing $g_n = \pi$,

- for $\tau \in S_n$ we have $\delta([g_{\tau(1)} \mid \cdots \mid g_{\tau(n-1)}]) = \mathrm{sign}(\tau)$.

We refer to such subgroups as Colmez subgroups. We define

$$\mathrm{Log} : \mathbb{R}^n_+ \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (\log(x_1), \dots, \log(x_n)).$$

Let $\mathcal{H} \subset \mathbb{R}^n$ be the hyperplane defined by $\mathrm{Tr}(z) = 0$. Then, $\mathrm{Log}(E_+(\mathfrak{f}))$ is a lattice in $\mathcal{H}$. If $z = (z_1, \dots, z_n) \in \mathbb{R}^n_+$ and $\mathrm{Log}(z) \in \mathbb{R}^n$ is not an element of $\mathcal{H}$, then we define the projection

$$z_{\mathcal{H}} = (z_1 \cdots z_n)^{-\frac{1}{n}} \cdot z.$$

We have $\mathrm{Log}(z_{\mathcal{H}}) \in \mathcal{H}$. Note that z and $z_{\mathcal{H}}$ lie on the same ray in $\mathbb{R}^n_+$. For any $M > 0$ and $i = 0, 1, \dots, n-1$, write $l_i(M)$ for the element of $\mathcal{H}$ which has value M in the $(i+1)$ place and $-M/(n-1)$ in the other places. We endow $\mathbb{R}^n$ with the sup-norm. We denote by $B(x, r)$ the ball centered at x of radius r.

The following lemma, which builds on [3, Lemma 2.1], allows us to find a collection of possible subsets $V = \langle g_1, \dots, g_{n-1}\rangle$ such that we get a nice sign property that allows us to more easily explicitly calculate the Eisenstein cocycle.

Lemma 4.3. *There exists $R_1 > 0$ such that for all $R > R_1$, $M > K_1(R)$ (where $K_1(R)$ is some constant we define that depends only on R) we have the following: for $i = 1, \dots, n-1$ let $g_i \in E_+(\mathfrak{f})$ and $g_n = g_\pi \in \pi_{\mathcal{H}} E_+(\mathfrak{f})$ such that $\mathrm{Log}(g_i) \in B(l_i(M), R)$ and $\mathrm{Log}(g_\pi) \in B(l_0(M), R)$. Then*

- $\langle g_1, \dots, g_{n-1}\rangle \subseteq E_+(\mathfrak{f})$ *is a finite index subgroup, and furthermore*
- *For $\tau \in S_n$ we have $\delta([g_{\tau(1)} \mid \cdots \mid g_{\tau(n-1)}]) = \mathrm{sign}(\tau)$.*

Proof. This proof largely follows the ideas of Colmez in his proof of [3, Lemma 2.1]. First, note that both $\mathrm{Log}(E_+(\mathfrak{f}))$ and $\mathrm{Log}(\pi_{\mathcal{H}} E_+(\mathfrak{f}))$ are lattices inside $\mathcal{H}$. There exists a constant $R_1 := R(E_+(\mathfrak{f}), \pi)$ such that for all $M > 0$ and any $r > R(E_+(\mathfrak{f}), \pi)$ there exist $g_1, \dots, g_{n-1} \in E_+(\mathfrak{f})$ and $g_\pi \in \pi_{\mathcal{H}} E_+(\mathfrak{f})$ such that $\mathrm{Log}(g_i) \in B(l_i(M), r)$ for $i = 1, \dots, n-1$ and $\mathrm{Log}(g_\pi) \in B(l_0(M), r)$. The existence of R_1 follows from Dirichlet's unit theorem and, in particular, the non-vanishing of the regulator of a

number field. Since the $l_i(M)$ form a basis of $\mathcal{H}$, the $\mathrm{Log}(g_i)$ form a free family of finite index in $\mathrm{Log}(E_+(\mathfrak{f}))$, if M is large enough relative to r, say $M > k(r)$. This gives the first point of the Lemma. It remains to show the second point.

Now, take M satisfying:

i) $M \geqslant 2(n-1)^4 r$,

ii) $M > (n-1)^2 \log(n!)$,

iii) $M > k(r)$.

For simplicity, let $K_1(r) = \max(2(n-1)^4 r, (n-1)^2 \log(n!), k(r))$ so that we only require $M > K_1(r)$.

Write $g_n = g_\pi$ and let $\tau \in S_n$. Denote $\Delta_\tau = \det([g_{\tau(1)} \mid \cdots \mid g_{\tau(n-1)}])$. We show $\Delta_\tau > 0$. If τ fixes n then this calculation is covered by the proof of [3, Lemma 2.1].

Suppose τ is the transposition which swaps $n-1$ and n. Then, $\Delta_\tau = \det([g_1 \mid \cdots \mid g_{n-2} \mid g_n])$. Put $E_i = \exp(M(1 - \frac{i-2}{n-1}))$ and $F_i = \exp(-M(\frac{i-1}{n-1}))$. Hence, the matrix given by $[g_1 \mid \cdots \mid g_{n-2} \mid g_n]$ is written as

$$\begin{pmatrix} 1 & \beta_{1,2}F_2 & \beta_{1,3}F_3 & \cdots & \beta_{1,n-1}F_{n-1} & \beta_{1,n}E_n \\ 1 & \beta_{2,2}E_2 & \beta_{2,3}E_3 & \cdots & \beta_{2,n-1}E_{n-1} & \beta_{2,n}E_n \\ 1 & \beta_{3,2}F_2 & \beta_{3,3}E_3 & \cdots & \beta_{2,n-1}E_{n-1} & \beta_{3,n}E_n \\ 1 & \beta_{4,2}F_2 & \beta_{4,3}F_3 & \cdots & \beta_{2,n-1}E_{n-1} & \beta_{3,n}E_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_{n-1,2}F_2 & \beta_{n-1,3}F_3 & \cdots & \beta_{n-1,n-1}E_{n-1} & \beta_{n-1,n}E_n \\ 1 & \beta_{n,2}F_2 & \beta_{n,3}F_3 & \cdots & \beta_{n,n-1}F_{n-1} & \beta_{n,n}F_n \end{pmatrix},$$

where by i),

$$e^{\frac{-M}{2(n-1)^3}} < \beta_{i,j} < e^{\frac{M}{2(n-1)^3}}.$$

Expand Δ_τ and isolate the term given by the entries with coordinates $(1, n), (2, 2), \ldots, (n-1, n-1), (n, 1)$. Using the bounds we defined previously we obtain

$$\mid \Delta_\tau - e^{\frac{nM}{2}} \beta_{1,n} \prod_{i=2}^{n-1} \beta_{i,i} \mid \leqslant (n! - 1)e^{\frac{M}{2(n-1)^2}} e^{M(\frac{n}{2} - \frac{n}{n-1})}$$

and so

$$\Delta_\tau \geqslant e^{\frac{nM}{2}} (e^{\frac{-M}{2(n-1)^2}} - (n! - 1)e^{(\frac{M}{2(n-1)^2} - \frac{nM}{n-1})}) > 0$$

according to ii). We then show the other required sign properties in the same way. □

It is required in our later calculations to make the following sign calculation.

Lemma 4.4. *For $i = 1, \ldots, n-1$ let $g_i \in E_+(\mathfrak{f})$ be chosen as in Lemma 4.3. Write S for the $n \times n$ matrix with rows $\mathrm{Log}(g_1), \ldots, \mathrm{Log}(g_{n-1}), v_0$ where $v_0 = (1, \ldots, 1) \in \mathbb{R}^n$. Then, if $M > 4(n! - 1)R$, we have*

$$\mathrm{sign}(\det(S)) = (-1)^{n-1}.$$

Proof. We have

$$
S = \begin{pmatrix}
-\frac{M}{n-1} + \beta_{1,1} & M + \beta_{1,2} & -\frac{M}{n-1} + \beta_{1,3} & \cdots & -\frac{M}{n-1} + \beta_{1,n} \\
-\frac{M}{n-1} + \beta_{2,1} & -\frac{M}{n-1} + \beta_{2,2} & M + \beta_{2,3} & \cdots & -\frac{M}{n-1} + \beta_{2,n} \\
-\frac{M}{n-1} + \beta_{3,1} & -\frac{M}{n-1} + \beta_{3,2} & -\frac{M}{n-1} + \beta_{3,3} & \cdots & -\frac{M}{n-1} + \beta_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{M}{n-1} + \beta_{n-1,1} & -\frac{M}{n-1} + \beta_{n-1,2} & -\frac{M}{n-1} + \beta_{n-1,3} & \cdots & M + \beta_{n-1,n} \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix},
$$

where $-R < \beta_{i,j} < R$. We now subtract the first column from each of the other columns and expand the determinant along the bottom row. This gives, after letting $B_{i,j} = \beta_{i,j} - \beta_{i,1}$,

$$
\det S = (-1)^{n-1} \det \begin{pmatrix}
\frac{nM}{n-1} + B_{1,2} & B_{1,3} & \cdots & B_{1,n} \\
B_{2,2} & \frac{nM}{n-1} + B_{2,3} & \cdots & B_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
B_{n-1,2} & B_{n-1,2} & \cdots & \frac{nM}{n-1} + B_{n-1,n}
\end{pmatrix}. \tag{18}
$$

Write S' for the matrix in (18) and note that $-2R < B_{i,j} < 2R$, for all $i, j = 1, \ldots, n-1$. When expanding the determinant of S' and isolating the diagonal terms using the bounds from before, we observe:

$$
\left| \det(S') - \prod_{i=1}^{n-1} \left(\frac{nM}{n-1} + B_{i,i+1} \right) \right| \leqslant (n! - 1) 2R \left(\frac{nM}{n-1} + 2R \right)^{n-2}.
$$

Thus,

$$
\det S' \geqslant \left(\frac{nM}{n-1} - 2R \right)^{n-1} - (n! - 1) 2R \left(\frac{nM}{n-1} + 2R \right)^{n-2}.
$$

Since we have assumed $M > 4(n! - 1)R$ we have

$$
\det S' > \left(\frac{nM}{n-1} - \frac{M}{2(n! - 1)} \right)^{n-1} - \frac{M}{2} \left(\frac{nM}{n-1} + \frac{M}{2(n! - 1)} \right)^{n-2}.
$$

It thus remains to show that for $n \geqslant 2$ the following holds:

$$
\left(\frac{n}{n-1} - \frac{1}{2(n! - 1)} \right)^{n-1} - \frac{1}{2} \left(\frac{n}{n-1} + \frac{1}{2(n! - 1)} \right)^{n-2} > 0. \tag{19}
$$

First, one can see by calculating that the inequality holds for $n = 2$. Remarking that as n increases the difference between the two terms in brackets decreases, gives that the value of the left hand side of (19) must increase with n. Thus, (19) holds. $\qquad \square$

We now let $K_2(R) = \max(K_1(R), 4(n! - 1)R)$ so that both Lemmas 4.3 and Lemma 4.4 hold if $M > K_2(R)$.

Corollary 4.5. *Let $r > 0$ be an integer, D_+ an $r \times r$ diagonal matrix with positive entries, $A \in M_{n \times r}(\mathbb{R})$ and S as in Lemma 4.4. Then, the block matrix*

$$B = \left(\begin{array}{c|c} A & D_+ \\ \hline S & 0 \end{array} \right),$$

has determinant of sign $(-1)^{n-1}(-1)^{r(n+r-1)}$.

Proof. Write $d_1, \dots, d_r \in \mathbb{R}_{>0}$ for the diagonal entries of D_+. Using cofactor expansion with the last r columns of B one can see that the determinant of B is equal to

$$\det(S) \prod_{i=1}^{r} d_i (-1)^{(n+r-i)+(i-1))} = \det(S)(-1)^{r(n+r-1)} \prod_{i=1}^{r} d_i.$$

Using Lemma 4.4 and the fact that the entries of D_+ are positive, the result follows. $\qquad \square$

We recall the definition of $k(r)$ from the proof of Lemma 4.3 and note the following lemma.

Lemma 4.6. *We can choose $k(r) = Kr$ where K is some constant that does not depend on r. That is, suppose $r > R_1$ and $M > Kr$, if for $i = 1, \dots, n-1$, we have $g_i \in E_+(\mathfrak{f})$ with $\mathrm{Log}(g_i) \in B(l_i(M), r)$ then the $\mathrm{Log}(g_i)$ form a free family of finite index in $\mathrm{Log}(E_+(\mathfrak{f}))$.*

Proof. We begin by noting that the result is trivial if $n = 2$. Suppose that $n > 2$. We claim that it is enough to take $K = 2(n-1)$. For each $i = 1, \dots, n-1$, let $\mathrm{Log}(g_i) \in B(l_i(M), r)$. We then write

$$\mathrm{Log}(g_i) = (\alpha_i(1), \dots, \alpha_i(n)) \in \mathcal{H}.$$

It is enough to show that the $\mathrm{Log}(g_i)$ are linearly independent under the projection

$$\varphi : \mathcal{H} \to \mathbb{R}^{n-1}$$

$$(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \alpha_{n-1}).$$

By the definition of $l_i(M)$ and our choice of r it is clear that

$$\alpha_i(j) > 0 \text{ if } j = i+1 \quad \text{and} \quad \alpha_i(j) < 0 \text{ otherwise.}$$

We note that $\alpha_{n-1}(j) < 0$ for all j. It follows immediately that the vectors

$$\varphi(\mathrm{Log}(g_1)), \dots, \varphi(\mathrm{Log}(g_{n-1}))$$

are linearly independent. Thus, the $\mathrm{Log}(g_i)$ for a free family of finite index in $\mathrm{Log}(E_+(\mathfrak{f}))$. $\qquad \square$

It follows from the above lemma that if $M > K_2(R)$ then for any $\lambda > 1$ we have that $\lambda M > K_2(\lambda R)$.

Lemma 4.7. *Let $R_1 > 0$ be as is shown to exist in Lemma 4.3. There exists*

1. $R_f, R_g > R_1$,
2. $M_f > K_2(R_f)$ and
3. $M_g > K_2(R_g)$,

such that we have the following. First, for $i = 1, \dots, n - 1$ we can choose $f_i, g_i \in E_+(\mathfrak{f})$ such that $\mathrm{Log}(f_i) \in B(l_i(M_f), R)$ *and* $\mathrm{Log}(g_i) \in B(l_i(M_g), R)$. *Furthermore, after writing*

$$V_f = \langle f_1, \dots, f_{n-1} \rangle \quad and \quad V_g = \langle g_1, \dots, g_{n-1} \rangle$$

we have that $[E_+(\mathfrak{f}) : V_f]$ *is coprime to* $[E_+(\mathfrak{f}) : V_g]$.

Proof. We first choose the $f_i \in E_+(\mathfrak{f})$ via Lemma 4.3 and Lemma 4.4, and let $V_f = \langle f_1, \dots, f_{n-1} \rangle$. That is, we have $\mathrm{Log}(f_i) \in B(l_i(M_f), R_f)$ for some $R_f > R_1$ and $M_f > K_2(R_f)$.

By writing the matrix representing the generators we have chosen for V_f in an upper triangular form, we can make the following choice of generators of $E_+(\mathfrak{f})$. Let $\langle \delta_1, \dots, \delta_{n-1} \rangle = E_+(\mathfrak{f})$ such that for some $\tau \in S_{n-1}$ we have, for $i = 1, \dots, n - 1$,

$$f_{\tau(i)} = \delta_i^{a_i} \prod_{j=1}^{i-1} \delta_j^{b_{i,j}},$$

and $[E_+(\mathfrak{f}) : V_f] = \prod_{i=1}^{n-1} |a_i|$. By changing the sign if necessary we choose $a_1 > 0$. Furthermore, we note that changing the values of the $b_{i,j}$ in the choice of V_f does not change the index of the subgroup.

For ease of notation, let $a = \prod_{i=1}^{n-1} |a_i|$. For $i = 2, \dots, n - 1$ there exists $R_{g,i} > 0$ and $M_{g,i} > 0$ such that for all $M > M_{g,i}$, there exists $\alpha \in E_+(\mathfrak{f})$ with $\mathrm{Log}(\alpha) \in B(l_{\tau(i)}(M), R_{g,i})$ and

$$\alpha = \delta_i^{q_i} \prod_{j=1}^{i-1} \delta_j^{k_j},$$

with q_i a nonzero integer with absolute value coprime to a. We note that this is only possible for $i > 2$ since we require the freedom of having at least one additional component we can vary.

We now consider $i = 1$. We have $\mathrm{Log}(f_{\tau(1)}) = \mathrm{Log}(\delta_1^{a_1}) \in B(l_1(M_f), R_f)$. Therefore, any $q_1 > a_1$ we have $\mathrm{Log}(\delta_1^{q_1}) \in B(\frac{q_1}{a_1} l_1(M_f), \frac{q_1}{a_1} R_f)$.

Now, let $R_g' = \max(R_1, R_{g,2}, \dots, R_{g,n-1})$ and $M_g' = \max(M_{g,2}, \dots, M_{g,n-1})$. We now find $q_1 > a_1$ which is coprime to a and such that $\frac{q_1}{a_1} M_f > M_g'$ and $\frac{q_1}{a_1} R_f > R_g'$.

We now fix $R_g = \frac{q_1}{a_1} R_f$ and $M_g = \frac{q_1}{a_1} M_f$. Clearly, $R_g > R_1$ and it follows from Lemma 4.6 that $M_g > K_2(R_g)$. We then choose $g_{\tau(1)} = \delta_1^{q_1}$, it is immediate that $\mathrm{Log}(g_{\tau(1)}) \in B(l_1(M_g), R_g)$. For $i = 2, \dots, n - 1$, we have shown that there exist $g_{\tau(i)} \in E_+(\mathfrak{f})$ with $\mathrm{Log}(g_{\tau(i)}) \in B(l_{\tau(i)}(M_g), R_g)$ and

$$g_{\tau(i)} = \delta_i^{q_i} \prod_{j=1}^{i-1} \delta_j^{k_j},$$

with q_i a nonzero integer with absolute value coprime to a. Let $V_g = \langle g_1, \dots, g_{n-1} \rangle$, the result follows. $\square$

4.5 | 1-Cocycles attached to homomorphisms

Let $g : F_{\mathfrak{p}}^* \to A$ be a continuous homomorphism, where A is a locally profinite group. We now define a cohomology class $c_g \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, A))$ attached to g. The $F_{\mathfrak{p}}^*$-action on $C_c(F_{\mathfrak{p}}^*, \mathbb{Z})$ is

defined by $(xf)(y) = f(x^{-1}y)$. The following definition is due to the third author and first appears in [16, Lemma 2.11]. This definition is crucial in making the constructions of the first and third author's and cohomological formulas' work. We also remark that the definition is unusual in that it appears as though the cocycle z_g should be a coboundary. However, it may not be a coboundary since g does not necessarily extend to a continuous function on $F_{\mathfrak{p}}$.

Definition 4.8. Let $g : F_{\mathfrak{p}}^* \to A$ be a continuous homomorphism, where A is a locally profinite group. Let $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ such that $f(0) = 1$. We define c_g to be the class of the cocycle

$$z_{f,g} : F_{\mathfrak{p}}^* \to C_c(F_{\mathfrak{p}}, A)$$

defined by $z_{f,g}(x) = \}\}(1 - x)(g \cdot f)''$, or more precisely

$$z_{f,g}(x)(y) = (xf)(y) \cdot g(x) + ((f - xf) \cdot g)(y) \tag{20}$$

for $x \in F_{\mathfrak{p}}^*$ and $y \in F_{\mathfrak{p}}$.

The second term in (20) is allowed to be evaluated at $0 \in F_{\mathfrak{p}}$ since we can extend continuously the function from $F_{\mathfrak{p}}^*$ to $F_{\mathfrak{p}}$ as

$$(f - xf)(0) = 0.$$

The class $c_g = [z_{f,g}] \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, A))$ is independent of the choice of $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ with $f(0) = 1$. In particular, we can consider the class $c_{\mathrm{id}} \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*))$. For more details on this construction, see [8, §3.2] and [9, §3.1].

4.6 | Homology of a group of units

Let $V \subseteq E_+$ be a finite index subgroup. Recall we have written $G_{\mathfrak{f}}$ for the narrow ray class group of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$, and write $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. Write $V_{\mathfrak{p}} = V \oplus \langle \pi \rangle$.

By Dirichlet's unit theorem, the group $V_{\mathfrak{p}}$ is free abelian of rank n. Thus, the homology groups $H_n(V_{\mathfrak{p}}, \mathbb{Z})$ are free abelian of rank 1. In the comological formulas u_2 and u_3, we are required to choose a generator of this homology group. For these two invariants, we will be working in the cases $V = E_+$ and $V = E_+(\mathfrak{f})$, respectively.

Write $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$. For ease of notation, write $\pi = \varepsilon_n$. We then choose the following generator for the group $H_n(V_{\mathfrak{p}}, \mathbb{Z})$,

$$\eta_{\mathfrak{p}} = \mu \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n)}] \otimes 1. \tag{21}$$

Here, $\mu \in \{1, -1\}$ and is equal to the sign of the determinant of a specific matrix. For $x \in E_{\mathfrak{p},+}$, let

$$L(x) = (\log(\sigma_1(x)), \dots, \log(\sigma_n(x)), \mathrm{ord}_{\mathfrak{p}}(x)).$$

Define $L_1 \in \mathbb{R}^n$ to be the vector with 1 in the first n components and 0 in the last component. Then, μ is the sign of the determinant of the matrix with rows

$$L(\pi), L(\varepsilon_1), \dots, L(\varepsilon_{n-1}), L_1.$$

This choice generalizes that given in [17, Remark 2.1].

5 | COHOMOLOGICAL FORMULA I (u_2)

This section follows the construction given in [8, §3.1]. For ease of notation and to reduce the exposition of this section, we give a simpler definition for u_2 than appears in [8, §3.1]. In particular, we do not involve the infinite places in the construction we give. For our purposes, this definition is enough and it simplifies the arguments in Section 7. Throughout this section, we use the notation established in Section 4.1.

Let $\eta_{\mathfrak{p}}$ be the generator of $H_n(E_{\mathfrak{p},+}, \mathbb{Z})$ defined in (21) and let $\mathcal{F}$ be a fundamental domain for the action of $F_+^\ell / E_{\mathfrak{p},+}$ on $(\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U^{\mathfrak{p},\ell,\infty}$. Then, $\mathbb{1}_{\mathcal{F}}$ is an element of $H^0(E_{\mathfrak{p},+}, C(\mathcal{F}, \mathbb{Z})) = C(\mathcal{F}, \mathbb{Z})^{E_{\mathfrak{p},+}}$. Taking the cap product gives $\mathbb{1}_{\mathcal{F}} \cap \eta_{\mathfrak{p}} \in H_n(E_{\mathfrak{p},+}, C(\mathcal{F}, \mathbb{Z}))$. We now define $\vartheta^{\mathfrak{p}} \in H_n(F^*, C_c(\emptyset, \mathbb{Z})^{\mathfrak{p},\ell,\infty})$ as the homology class corresponding to $\mathbb{1}_{\mathcal{F}} \cap \eta_{\mathfrak{p}}$ under the isomorphism

$$H_n(E_{\mathfrak{p},+}, C(\mathcal{F}, \mathbb{Z})) \cong H_n(F_+^\ell, C_c((\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U^{\mathfrak{p},\ell,\infty}, \mathbb{Z})) \tag{22}$$

that is induced by $C_c((\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U^{\mathfrak{p},\ell,\infty}, \mathbb{Z}) \cong \mathrm{Ind}_{E_{\mathfrak{p},+}}^{F_+^\ell} C(\mathcal{F}, \mathbb{Z})$.

We now follow the construction of [8, §6]. Since the local norm residue symbol for H/F at $\mathfrak{p}$ is trivial we omit it from the reciprocity map, that is, we consider the homomorphism

$$\mathrm{rec}_{H/F}^{\mathfrak{p},\ell,\infty} : (\mathbb{A}_F^{\mathfrak{p}})^* / U^{R,\mathfrak{p},\ell} \to G \hookrightarrow \mathbb{Z}[G]^*, \quad x = (x_v)_{v \neq \mathfrak{p}} \mapsto \prod_{v \notin \{\mathfrak{p},\ell\} \cup R} (x, H/F)_v.$$

Let $R' = R - R_\infty$. We can view $\mathrm{rec}_{H/F}^{\mathfrak{p},\ell,\infty}$ as an element of $H^0(F_+^\ell, C_c(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty})$ and denote by

$$\rho_{H/F} \in H_n(F_+^\ell, C_c(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty})$$

its image under the map

$$H^0(F_+^\ell, C_c(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty}) \to H_n(F_+^\ell, C_c(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty}), \quad \psi \mapsto \psi \cap \vartheta^{\mathfrak{p}}.$$

Here, the cap product is induced by the map

$$C^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty} \times C_c(\emptyset, \mathbb{Z})^{\mathfrak{p}} \to C_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty}, \quad (\psi, \phi) \mapsto \psi \cdot \phi, \tag{23}$$

here $\psi \cdot \phi$ denotes the function $xU^{R' \cup \{\mathfrak{p},\ell,\infty\}} \mapsto \psi(xU^{R' \cup \{\mathfrak{p},\ell,\infty\}})\phi(xU^{\mathfrak{p}})$.

For a locally profinite abelian group A, we have a canonical map

$$C_c^\diamond(F_{\mathfrak{p}}, A) \otimes C_c(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty} \to C_c^\diamond(\mathfrak{p}, R', A \otimes \mathbb{Z}[G])^{\ell,\infty}, \quad (f, g) \mapsto f \otimes g,$$

which induces a cap-product pairing

$$H^1(F^*, C_c^\diamond(F_{\mathfrak{p}}, A)) \times H_n(F^*, C_c(R', \mathbb{Z}[G])^{\mathfrak{p},\ell,\infty}) \to H_{n-1}(F^*, C_c^\diamond(\mathfrak{p}, R', A \otimes \mathbb{Z}[G])^{\ell,\infty}).$$

In particular, we can consider

$$c_{\mathrm{id}} \cap \rho_{H/F} \in H_{n-1}(F^*, C_c^\diamond(\mathfrak{p}, R', F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])^{\ell,\infty}).$$

Here, c_{id} is as defined in Definition 4.8. Recall that we write W for F considered as a $\mathbb{Q}$-vector space. In [8, §5.3], the following map is defined as

$$\Delta_* : H_{n-1}(F_+^\ell, C_c^\diamond(\mathfrak{p}, R', F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])^{\ell,\infty}) \to H_{n-1}(F_+^\ell, C_c^\diamond(W_{\widehat{\mathbb{Z}}^\ell}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])).$$

We postpone giving the definition of Δ_* until the next section.

Now consider the canonical pairing, where we recall the definition of $\mu_{F_{\mathfrak{p}}^*}$ from Section 4.2,

$$\mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\ell}, \mathbb{Z}), \mathbb{Z}) \times C_c^\diamond(W_{\widehat{\mathbb{Z}}^\ell}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]) \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G], \quad (\mu, f) \mapsto \mu_{F_{\mathfrak{p}}^*}(f). \qquad (24)$$

Noting that F_+^ℓ is acting trivially on $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ we see that (24) induces, via cap-product, a pairing

$$\cap : H^{n-1}(F_+^\ell, \mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\ell}, \mathbb{Z}), \mathbb{Z})) \times H_{n-1}(F_+^\ell, C_c^\diamond(W_{\widehat{\mathbb{Z}}^\ell}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])) \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]. \qquad (25)$$

Recall the Eisenstein cocycle, $\mathrm{Eis}_{F,\lambda}^0$, from Proposition 4.2. Applying (25) with the Eisenstein cocycle $\mathrm{Eis}_{F,\lambda}^0$ and $\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F})$ we obtain the element of $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$, defined in [8, §3.1]. Therefore,

$$u_2 = u(S, \lambda) = \sum_{\sigma \in G} u_2(\sigma) \otimes [\sigma^{-1}] = \mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H/F}). \qquad (26)$$

The first and third authors then conjecture that the element $u_2(\sigma)$ is equal to the image of the Brumer–Stark unit in $F_{\mathfrak{p}}^*$ under σ. We end this section by stating some known properties of this construction.

Remark 5.1. As noted at the start of this section, the definition of u_2 given above is equivalent to that given in [8, §3.1]. This follows from standard properties of the cap-product.

Proposition 5.2 (Proposition 6.3, [8]).

a) *For $\sigma \in G$ we have $\mathrm{ord}_{\mathfrak{p}}(u_2(\sigma)) = \zeta_{R,T}(\sigma, 0)$.*

b) *Let L/F be an abelian extension with $L \supseteq H$ and put $\mathfrak{g} = \mathrm{Gal}(L/F)$. Assume that L/F is unramified outside S and that $\mathfrak{p}$ splits completely in L. Then, we have*

$$u_2(\sigma) = \prod_{\tau \in \mathfrak{g}, \tau|_H = \sigma} u_2(L/F, \tau).$$

c) *Let $\mathfrak{r}$ be a nonarchimedean place of F with $\mathfrak{r} \notin S \cup \overline{T}$ where $\overline{T}$ is as defined in (4). Then, we have*

$$u_2(S \cup \{\mathfrak{r}\}, \sigma) = u_2(S, \sigma) u_2(S, \sigma_{\mathfrak{r}}^{-1}\sigma)^{-1}.$$

d) *Assume that H has a real Archimedean place. Then, $u_2(\sigma) = 1$ for all $\sigma \in G$.*

e) *Let L/F be a finite abelian extension of F containing H and unramified outside S. Then, we have*

$$\mathrm{rec}_{\mathfrak{p}}(u_2(\sigma)) = \prod_{\substack{\tau \in \mathrm{Gal}(L/F) \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,T}(L/F, \tau^{-1}, 0)}.$$

Remark 5.3. In the proposition above, we correct a small typo in [8, Proposition 6.3, c)] by replacing $\sigma_{\mathfrak{r}}$ with $\sigma_{\mathfrak{r}}^{-1}$.

5.1 | The map Δ_*

We now define the map Δ_*. For more information and the more general construction, we refer to [8, §5.3]. Throughout this section, we let $A = F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ to ease notation. For sets X_1, X_2 and a map $\psi : X_1 \times X_2 \to A$, we write

$$\mathrm{Supp}(X_1, X_2, \psi) := \{x_1 \in X_1 \mid \exists\, x_2 \in X_2 \text{ with } (x_1, x_2) \in \mathrm{supp}(\psi)\},$$

where $\mathrm{supp}(\psi)$ is the support of ψ.

Proposition 5.4. *Let X_1, X_2 be totally disconnected topological Hausdorff spaces, with X_1 discrete. Let A be a locally profinite group. The map*

$$C_c(X_1, \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^{\diamond}(X_2, A) \to C_c^{\diamond}(X_1 \times X_2, A), \tag{27}$$

$$f \otimes g \mapsto ((x_1, x_2) \mapsto f(x_1) \cdot g(x_2))$$

is an isomorphism.

Proof. We calculate the inverse map as follows. For $\psi \in C_c^{\diamond}(X_1 \times X_2, A)$, we write $Y_1(\psi) = \mathrm{Supp}(X_1, X_2, \psi) \subseteq X_1$. Note that $Y_1(\psi)$ is finite since ψ has compact support. Then

$$\psi \mapsto \sum_{y \in Y_1(\psi)} \mathbb{1}_y \otimes_{\mathbb{Z}} \psi(y, \cdot) \in C_c(X_1, \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^{\diamond}(X_2, A)$$

provides an inverse to (27). $\qquad\square$

We now construct the F_+^{ℓ}-equivariant map

$$\Delta : C_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\ell, \infty} \to C_c^{\diamond}(\mathbb{A}_F^{\ell, \infty}, A) \cong C_c^{\diamond}(W_{\widehat{\mathbb{Z}}^{\ell}}, A). \tag{28}$$

Recall that we have written $S' = R' \cup \{\mathfrak{p}\}$ and $\mathbb{A}_F^{\ell, \infty} \cong W_{\widehat{\mathbb{Z}}^{\ell}}$. There exist canonical homomorphisms

$$C_c^{\diamond}\left(F_{\mathfrak{p}} \times \prod_{\mathfrak{q} \in R'} F_{\mathfrak{q}}^*, A\right) \otimes C_c(\emptyset, \mathbb{Z})^{S' \cup \ell, \infty} \to C_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\ell, \infty}, \tag{29}$$

$$C_c^{\diamond}\left(\prod_{\mathfrak{q} \in S'} F_{\mathfrak{q}}, A\right) \otimes C_c(\mathbb{A}_F^{S' \cup \ell, \infty}, \mathbb{Z}) \to C_c^{\diamond}(\mathbb{A}_F^{\ell, \infty}, A). \tag{30}$$

It follows from Proposition 5.4 that the map (29) is an isomorphism. Let $\mathcal{I}^{S'\cup\ell}$ denote the group of fractional ideals of F that are coprime to $S'\cup\ell$. Since $(\mathbb{A}_F^{S'\cup\ell,\infty})^*/U^{S'\cup\ell,\infty}$ is isomorphic to $\mathcal{I}^{S'\cup\ell}$, the ring $C_c^0(\emptyset,\mathbb{Z})^{S'\cup\ell,\infty}$ can be identified with the group ring $\mathbb{Z}[\mathcal{I}^{S'\cup\ell}]$. We define (28) as the tensor product $\Delta = i \otimes I^{S\cup\ell}$ where $i : C_c^\diamond(F_\mathfrak{p} \times \prod_{\mathfrak{q}\in R'} F_\mathfrak{q}^*, A) \to C_c^\diamond(\prod_{\mathfrak{q}\in S'} F_\mathfrak{q}, A)$ is the inclusion map induced by extension by 0 and $I^{S'\cup\ell} : \mathbb{Z}[\mathcal{I}^{S'\cup\ell}] \to C_c(\mathbb{A}_F^{S'\cup\ell,\infty},\mathbb{Z})$ maps a fractional ideal $\mathfrak{a} \in \mathcal{I}^{S'\cup\ell}$ to the characteristic function of $\widehat{\mathfrak{a}}^{S'\cup\ell} = \mathfrak{a}(\prod_{\mathfrak{p}\notin S'\cup\ell} \mathcal{O}_\mathfrak{p})$. Considering the map in (30) completes our construction of Δ.

6 │ COHOMOLOGICAL FORMULA II (u_3)

In [9], the first and third authors give two equivalent constructions for their formula. In this work, we only require the construction given in [9, §3.3], which we denote by u_3. We refer readers to [9, §3] for the other formula.

Recall that in Sections 4.5 and 4.3 we have defined the following objects:

$$c_{\mathrm{id}} \in H^1(F_\mathfrak{p}^*, C_c(F_\mathfrak{p}, F_\mathfrak{p}^*)) \quad \text{and} \quad \omega_{\mathfrak{f},\lambda}^\mathfrak{p} \in H^{n-1}(E_+(\mathfrak{f})_\mathfrak{p}, \mathrm{Hom}(C_c(F_\mathfrak{p},\mathbb{Z}),\mathbb{Z}[G])).$$

Definition 6.1. Let $\eta_{\mathfrak{p},E_+(\mathfrak{f})} \in H_n(E_+(\mathfrak{f})_\mathfrak{p}, \mathbb{Z})$ be the generator defined in (21). Then, we define

$$u_3 := (-1)^{n+1}(c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\lambda}^\mathfrak{p} \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})})) \in F_\mathfrak{p}^* \otimes \mathbb{Z}[G]. \tag{31}$$

As noted in the introduction, we have modified the definition from [9] by multiplying by $(-1)^{n+1}$, namely, if we let u_3' be the element defined in [9] then $u_3 = (-1)^{n+1}u_3'$. Adapted from [9, Conjecture 3.1] we have the following conjecture.

Conjecture 6.2. *We have $u_3 = u_\mathfrak{p}$.*

6.1 │ Transferring to a subgroup

Let V be a finite index subgroup of $E_+(\mathfrak{f})$ and $\mathfrak{b}$ a fractional ideal coprime to S and ℓ. Let $\eta_{\mathfrak{p},V} \in H_n(V \oplus \langle\pi\rangle, \mathbb{Z})$ be the generator defined in (21). For $x_1, \dots, x_n \in V \oplus \langle\pi\rangle$ and compact open $U \subset F_\mathfrak{p}$ we put

$$\nu_{\mathfrak{b},\lambda,V}^\mathfrak{p}(x_1,\dots,x_n)(U) := \delta(x_1,\dots,x_n)\zeta_{R,\lambda}(\mathfrak{b},\overline{C}_{e_1}(x_1,\dots,x_n),U,0).$$

As before, it follows from [2, Theorem 2.6] that $\nu_{\mathfrak{b},\lambda,V}^\mathfrak{p}$ is a homogeneous $(n-1)$-cocycle on $V \oplus \langle\pi\rangle$ with values in the space of the $\mathbb{Z}$-distribution on $F_\mathfrak{p}$. Hence, we obtain a class

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^\mathfrak{p} := [\nu_{\mathfrak{b},\lambda,V}^\mathfrak{p}] \in H^{n-1}(V \oplus \langle\pi\rangle, \mathrm{Hom}(C_c(F_\mathfrak{p},\mathbb{Z}),\mathbb{Z})).$$

We then define

$$u_3(V,\sigma_\mathfrak{b}) = (-1)^{n+1}(c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^\mathfrak{p} \cap \eta_{\mathfrak{p},V})),$$

and write

$$u_3(V) = \sum_{\sigma \in G} u_3(V, \sigma) \otimes \sigma \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G].$$

The next proposition shows the relation between u_3 and $u_3(V)$. Here, we adopt the convention that for an element $x = \Sigma_{\sigma \in G} x_\sigma \otimes \sigma \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ and $k \in \mathbb{Z}$ we write

$$x^k = \sum_{\sigma \in G} x_\sigma^k \otimes \sigma \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G].$$

Proposition 6.3 (Proposition 6.12, [14]). *Let V be a finite index subgroup of $E_+(\mathfrak{f})$. Then, we have*

$$u_3(V) = u_3^{[E_+(\mathfrak{f}):V]}. \tag{32}$$

6.2 │ Explicit expression for u_3

For later calculations, we require an explicit expression for $u_3(V)$ for an appropriate choice of $V \subset E_+(\mathfrak{f})$. Let V be a finite index subgroup of $E_+(\mathfrak{f})$ such that $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$, where $\varepsilon_1, \dots, \varepsilon_{n-1}$ and $\pi = \varepsilon_n$ are chosen to satisfy Lemmas 4.3 and 4.4. For $i = 1, \dots, n$ write

$$\mathcal{B}_i := \bigcup_{\substack{\tau \in S_n \\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]).$$

Let $\mathcal{B} = \mathcal{B}_n$.

Lemma 6.4. *Let V be a finite index subgroup of $E_+(\mathfrak{f})$ such that $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$, where $\varepsilon_1, \dots, \varepsilon_{n-1}$ and $\pi = \varepsilon_n$ are chosen to satisfy Lemmas 4.3 and 4.4. Then, for $\sigma \in G$, we have*

$$u_3(V, \sigma) = \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}_i, \pi \mathcal{O}_{\mathfrak{p}}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, \mathcal{O}_{\mathfrak{p}}, 0)} \oint_{\mathbb{O}} x \, d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0))(x). \tag{33}$$

Here, for $i = 1, \dots, n$, we write

$$\mathcal{B}_i := \bigcup_{\substack{\tau \in S_n \\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}])$$

and let $\mathcal{B} = \mathcal{B}_n$.

Proof. Recall we write $V_{\mathfrak{p}} = V \oplus \langle \pi \rangle$ and, as in (21), choose the following generator for $H_n(V_{\mathfrak{p}}, \mathbb{Z})$,

$$\eta_{\mathfrak{p}, V} = (-1) \sum_{\tau \in S_n} \mathrm{sign}(\tau) [\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n)}] \otimes 1.$$

Let $\sigma \in G$ and $\mathfrak{b}$ a fractional ideal coprime to S and ℓ such that $\sigma = \sigma_{\mathfrak{b}}$. Recall,

$$u_3(V, \sigma) = (-1)^{n+1}(c_{\mathrm{id}} \cap (\omega_{\mathfrak{f}, \mathfrak{b}, \lambda, V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p}, V})).$$

We calculate

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p},V} = (-1)^{n+1} \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \operatorname{sign}(\tau) \omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]) \otimes [\varepsilon_i].$$

We recall the definition of $\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}}$ from Section 6.1. For $\tau \in S_n$ and a compact open $U \subseteq \mathcal{O}_{\mathfrak{p}}$, we have

$$\operatorname{sign}(\tau) \omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]) = \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]), U, 0). \tag{34}$$

Returning to our main calculation, using (34) we have

$$c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p},V}) = (-1)^{n+1} \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \int_{F_{\mathfrak{p}}} z_{\mathrm{id}}(\varepsilon_i)(x) \, d(\varepsilon_i \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]), x, 0))$$

$$= (-1)^{n+1} \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \int_{F_{\mathfrak{p}}} \varepsilon_i^{-1} z_{\mathrm{id}}(\varepsilon_i)(x) \, d(\zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]), x, 0)).$$

We note that taking the cap product gives another factor of (-1) which cancels the factor from before. One can easily compute, as is done in the proof of [9, Proposition 4.6], that for $i = 1, \dots, n + r - 1, i \neq n$

$$\varepsilon_i^{-1} z_{\mathrm{id}}(\varepsilon_i) = \mathbb{1}_{\pi \mathcal{O}_{\mathfrak{p}}} \cdot \varepsilon_i, \tag{35}$$

and

$$\pi^{-1} z_{\mathrm{id}}(\pi) = \mathbb{1}_{\mathbb{O}} \cdot \mathrm{id}_{F_{\mathfrak{p}}^*} + \mathbb{1}_{\mathcal{O}_{\mathfrak{p}}} \cdot \pi. \tag{36}$$

Here, we recall the $F_{\mathfrak{p}}^*$-action on $C_c(F_{\mathfrak{p}}^*, \mathbb{Z})$ from Section 4.5. Applying (35) and (36) and piecing together the appropriate Shintani sets we further deduce

$$c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p},V}) = (-1)^{n+1} \fint_{\mathbb{O}} x \, d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0)) \fint_{\mathcal{O}_{\mathfrak{p}}} \pi \, d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0))$$

$$\prod_{i=1}^{n-1} \fint_{\pi \mathcal{O}_{\mathfrak{p}}} \varepsilon_i \, d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}_i, x, 0)). \tag{37}$$

It is clear that we can then write

$$u_3(V, \sigma) = (-1)^{n+1}(c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p},V}))$$

$$= \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b},\mathcal{B}_i,\pi \mathcal{O}_{\mathfrak{p}},0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathcal{B},\mathcal{O}_{\mathfrak{p}},0)} \fint_{\mathbb{O}} x \, d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0))(x). \qquad \square$$

7 | EQUALITY OF u_2 AND u_3

In this section, we prove the following theorem.

Theorem 7.1. *We have* $u_2 = u_3$.

In order to prove the above theorem we require the following lemma which records useful functorial properties of the cap-product.

Lemma 7.2. *Let G be a group, $H \subset G$ a subgroup and let $\iota : H \hookrightarrow G$ denote the inclusion map. Let A, B, C be G-modules, A', B', C' be H-modules and suppose*

$$\epsilon : A \times B \to C, \quad (\text{resp. } \epsilon' : A' \times B' \to C')$$

is a G-equivariant (resp. H-equrivariant) pairing inducing the cap-product pairing

$$\cap = \cap_\epsilon : H^i(G, A) \times H_j(G, B) \to H_{j-i}(G, C)$$

$$(\text{resp. } \cap = \cap_{\epsilon'} : H^i(H, A') \times H_j(H, B') \to H_{j-i}(H, C')).$$

Let $\alpha : A \to A'$, $\beta : B' \to B$ and $\gamma : C' \to C$ be H-equivariant maps such that

$$\epsilon(a, \beta(b')) = \gamma(\epsilon'(\alpha(a), b')) \quad \forall a \in A, b' \in B'.$$

The maps ι, α (resp. ι, β) induce a homomorphism

$$r_H^G(\alpha) = \alpha_* \circ res_H^G : H^i(G, A) \to H^i(H, A')$$

$$(\text{resp. } c_H^G(\beta) = cor_H^G \circ \beta_* : H_j(G, B') \to H_j(H, B)).$$

Then for $a \in H^i(G, A)$ and $b' \in H_j(H, B')$ the following formula holds:

$$a \cap_\epsilon c_H^G(\beta)(b') = c_H^G(\gamma)(r_H^G(\alpha)(a) \cap_{\epsilon'} b').$$

We modify the map Δ_* by omitting the place $\mathfrak{p}$ as well to obtain a F_+^ℓ-equivariant map

$$\Delta^\mathfrak{p} : C_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}, \ell, \infty} \to C_c^0(\mathbb{A}_F^{\mathfrak{p}, \ell, \infty}, \mathbb{Z}[G]).$$

As before, the map $\Delta^\mathfrak{p}$ induces a homomorphism

$$\Delta_*^\mathfrak{p} : H_n(F_+^\ell, C_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}, \ell, \infty}) \to H_n(F_+^\ell, C_c^0(\mathbb{A}_F^{\mathfrak{p}, \ell, \infty}, \mathbb{Z}[G])).$$

The natural pairing

$$C_c(F_\mathfrak{p}, F_\mathfrak{p}^*) \times C_c^0(\mathbb{A}_F^{\mathfrak{p}, \ell, \infty}, \mathbb{Z}[G]) \to C_c^0(\mathbb{A}_F^{\ell, \infty}, F_\mathfrak{p}^* \otimes \mathbb{Z}[G])$$

induces a cap-product pairing

$$H^1(F_+^\ell, C_c(F_\mathfrak{p}, F_\mathfrak{p}^*)) \times H_n(F_+^\ell, C_c^0(\mathbb{A}_F^{\mathfrak{p}, \ell, \infty}, \mathbb{Z}[G])) \to H_{n-1}(F_+^\ell, C_c^0(\mathbb{A}_F^{\ell, \infty}, F_\mathfrak{p}^* \otimes \mathbb{Z}[G])).$$

It is straightforward to see that we then have the equality

$$u_2 = \mathrm{Eis}^0_{F,\lambda} \cap (c_{\mathrm{id}} \cap \Delta^{\mathfrak{p}}_*(\rho_{H/F})). \tag{38}$$

We are now ready to prove the main theorem of this section.

Proof of Theorem 7.1. Let $\mathcal{F} \subseteq (\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}$ be a fundamental domain for the action of $F^\ell_+ / E_+(\mathfrak{f})_{\mathfrak{p}}$. Consider the $E_+(\mathfrak{f})_{\mathfrak{p}}$-equivariant map

$$j : \mathbb{Z} \to C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}), \quad 1 \mapsto \mathbb{1}_{\mathcal{F}}.$$

Let $\eta_{\mathfrak{p},E_+(\mathfrak{f})} \in H_n(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z})$ be the generator defined in (21) and let $\vartheta^{\mathfrak{p}}_{\mathfrak{f}}$ be the image of the class $\eta_{\mathfrak{p},E_+(\mathfrak{f})}$ under the map

$$c^{F^\ell_+}_{E_+(\mathfrak{f})_{\mathfrak{p}}} : H_n(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z}) \to H_n(F^\ell_+, C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z})).$$

Here, this map is induced by the map j, defined above. Let $\pi : C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}) \to C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}, \mathbb{Z})$ be the natural projection and write

$$\iota : C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}, \mathbb{Z}) \to C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}), \quad \varphi \mapsto \varphi \circ \pi$$

for the induced map. Using the fact that $\mathrm{cor}^{E_{\mathfrak{p},+}}_{E_+(\mathfrak{f})_{\mathfrak{p}}}(\eta_{\mathfrak{p}}) = \eta_{\mathfrak{p},E_+(\mathfrak{f})}$ one can observe that

$$\iota_*(\vartheta^{\mathfrak{p}}) = \vartheta^{\mathfrak{p}}_{\mathfrak{f}}, \tag{39}$$

(for more details we refer to the proof of [12, Lemma 5.1]). Note that the reciprocity map $\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F}$ factors through $(\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}$, to distinguish it from $\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F}$ we write

$$\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}} : (\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}} \to G \hookrightarrow \mathbb{Z}[G]^*.$$

As before we can view $\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}}$ as an element of $H^0(F^\ell_+, C((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}[G]))$. By (39), we have the equality

$$\Delta^{\mathfrak{p}}_*(\rho_{H/F}) = \Delta^{\mathfrak{p}}_*(\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}} \cap \vartheta^{\mathfrak{p}}_{\mathfrak{f}}). \tag{40}$$

Here, the cap-product is induced by the pairing

$$C((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}[G]) \times C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}) \to C((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}[G]),$$

$$(\phi, \psi) \mapsto \phi \odot \psi.$$

For any $b \in (\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^*$ we define maps

$$j_b : \mathbb{Z} \to C_c((\mathbb{A}^{\mathfrak{p},\ell,\infty}_F)^* / U^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}, \mathbb{Z}), \quad 1 \mapsto \mathbb{1}_{bU^{\mathfrak{p},\ell,\infty}_{\mathfrak{f}}}$$

$$\mathrm{ev}_b \ : \ C((\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}, \mathbb{Z}[G]) \to \mathbb{Z}[G], \quad \varphi \mapsto \varphi(bU_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}).$$

Note that for every $m \in \mathbb{Z}$ and $\varphi \in C((\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}, \mathbb{Z}[G])$ we have

$$j_b(m) \odot \varphi = j_b(m \cdot \mathrm{ev}_b(\varphi)). \tag{41}$$

Let $\vartheta_b^{\mathfrak{p}} \in H_n(F_+^{\ell}, C_c((\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}, \mathbb{Z})$ be the image of $\eta_{\mathfrak{p}, E_+(\mathfrak{f})}$ under the map

$$c_{E_+(\mathfrak{f})_{\mathfrak{p}}}^{F_+^{\ell}}(j_b) \ : \ H_n(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z}) \to H_n(F_+^{\ell}, C_c((\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}, \mathbb{Z})).$$

For the fundamental domain $\mathcal{F} = \{b_1 U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}, \dots, b_s U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty}\}$ we choose $b_1, \dots b_s \in (\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^*$ such that the b_i are coprime to $\mathfrak{f}$. We then write $\mathfrak{b}_1, \dots \mathfrak{b}_s$ for the corresponding fractional ideals of $\mathcal{O}_F$ and note that, by our choice of $b_1, \dots, b_s$, the ideals $\mathfrak{b}_1, \dots, \mathfrak{b}_s$ are coprime to $\mathfrak{f} \cdot \ell \cdot \mathfrak{p}$. Since

$$(\mathbb{A}_F^{\mathfrak{p},\ell,\infty})^* / F_+^{\ell} U_{\mathfrak{f}}^{\mathfrak{p},\ell,\infty} \cong G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle,$$

the collection of fractional ideals $\mathfrak{b}_1, \dots, \mathfrak{b}_s$ is a system of representatives of $G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle$. It follows that

$$\vartheta_{\mathfrak{f}}^{\mathfrak{p}} = \sum_{i=1}^{s} \vartheta_{b_i}^{\mathfrak{p}}.$$

Hence, by (40) we can calculate

$$\Delta_*^{\mathfrak{p}}(\rho_{H/F}) = \sum_{i=1}^{s} \Delta_*^{\mathfrak{p}}\left(\mathrm{rec}_{H/F,\mathfrak{f}}^{\mathfrak{p},\ell,\infty} \cap \vartheta_{b_i}^{\mathfrak{p}}\right). \tag{42}$$

For every $i \in \{1, \dots, s\}$ we have, by Lemma 7.2 and (41),

$$\mathrm{rec}_{H/F,\mathfrak{f}}^{\mathfrak{p},\ell,\infty} \cap \vartheta_{b_i}^{\mathfrak{p}} = c_{E_+(\mathfrak{f})_{\mathfrak{p}}}^{F_+^{\ell}}(j_{b_i})(\mathrm{rec}_{H/F}(\mathfrak{b}_i) \otimes \eta_{\mathfrak{p}, E_+(\mathfrak{f})}) = c_{E_+(\mathfrak{f})_{\mathfrak{p}}}^{F_+^{\ell}}(j_{b_i})(\eta_{\mathfrak{p}, E_+(\mathfrak{f})}) \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i). \tag{43}$$

Under the composition $\Delta^{\mathfrak{p}} \circ j_{b_i} \ : \ \mathbb{Z} \to C_c^0(\mathbb{A}_F^{\mathfrak{p},\ell,\infty}, \mathbb{Z})$ we have that 1 is mapped to the characteristic function of $\widehat{\mathfrak{b}_i}^{\mathfrak{p},\ell} := \mathfrak{b}_i \prod_{v \nmid \mathfrak{p},\ell,\infty}' \mathcal{O}_v$. We define the map

$$\delta_{\mathfrak{b}_i} \ : \ \mathrm{Hom}(C_c^0(\mathbb{A}_F^{\ell,\infty}, \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z})$$

by $\delta_{\mathfrak{b}_i}(\mu)(f) = \mu(f \otimes \mathbb{1}_{\widehat{\mathfrak{b}_i}^{\mathfrak{p},\ell}})$ for $\mu \in \mathrm{Hom}(C_c^0(\mathbb{A}_F^{\ell,\infty}, \mathbb{Z}), \mathbb{Z})$ and $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$. It then follows from the definitions of $\mathrm{Eis}_{F,\lambda}^0$ and $\omega_{\mathfrak{f},\mathfrak{b}_i,\lambda}$ that

$$r_{E_+(\mathfrak{f})_{\mathfrak{p}}}^{F_+^{\ell}}(\delta_{\mathfrak{b}_i})(\mathrm{Eis}_{F,\lambda}^0) = \omega_{\mathfrak{f},\mathfrak{b}_i,\lambda}. \tag{44}$$

Here

$$r_{E_+(\mathfrak{f})_{\mathfrak{p}}}^{F_+^{\ell}}(\delta_{\mathfrak{b}_i}) \ : \ H^{n-1}(F_+^{\ell}, \mathrm{Hom}(C_c^0(\mathbb{A}_F^{\ell,\infty}, \mathbb{Z}), \mathbb{Z})) \to H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z}))$$

is the induced map, as defined in Lemma 7.2. We calculate, using standard properties of the cap-product and (43), that

$$\mathrm{Eis}^0_{F,\lambda} \cap (c_{\mathrm{id}} \cap \Delta^{\mathfrak{p}}_*(\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}} \cap \vartheta^{\mathfrak{p}}_{b_i})) = (-1)^{n-1} c_{\mathrm{id}} \cap (\mathrm{Eis}^0_{F,\lambda} \cap \Delta^{\mathfrak{p}}_*(\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}} \cap \vartheta^{\mathfrak{p}}_{b_i}))$$

$$= (-1)^{n-1} c_{\mathrm{id}} \cap (\mathrm{Eis}^0_{F,\lambda} \cap \Delta^{\mathfrak{p}}_*(c^{F^{\ell}_+}_{E_+(\mathfrak{f})_{\mathfrak{p}}}(j_{b_i})(\eta_{\mathfrak{p},E_+(\mathfrak{f})}) \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i)))$$

$$= (-1)^{n-1} c_{\mathrm{id}} \cap (\mathrm{Eis}^0_{F,\lambda} \cap \Delta^{\mathfrak{p}}_*(c^{F^{\ell}_+}_{E_+(\mathfrak{f})_{\mathfrak{p}}}(j_{b_i})(\eta_{\mathfrak{p},E_+(\mathfrak{f})}))) \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i)$$

$$= (-1)^{n-1} c_{\mathrm{id}} \cap (\mathrm{Eis}^0_{F,\lambda} \cap c^{F^{\ell}_+}_{E_+(\mathfrak{f})_{\mathfrak{p}}}(\Delta^{\mathfrak{p}} \circ j_{b_i})(\eta_{\mathfrak{p},E_+(\mathfrak{f})})) \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i).$$

Applying Lemma 7.2 and the equality (44) to the above calculation then yields,

$$\mathrm{Eis}^0_{F,\lambda} \cap (c_{\mathrm{id}} \cap \Delta^{\mathfrak{p}}_*(\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}} \cap \vartheta^{\mathfrak{p}}_{b_i})) = (-1)^{n-1} c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b}_i,\lambda} \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})}) \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i)$$

$$= (-1)^{n-1} c_{\mathrm{id}} \cap ((\omega_{\mathfrak{f},\mathfrak{b}_i,\lambda} \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i)) \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})}). \qquad (45)$$

Recalling the alternative definition of u_2 from (38) and the equality (42) we have

$$u_2 = \mathrm{Eis}^0_{F,\lambda} \cap (c_{\mathrm{id}} \cap \Delta^{\mathfrak{p}}_*(\rho_{H/F})) = \sum_{i=1}^s (-1)^{n-1} (\mathrm{Eis}^0_{F,\lambda} \cap (c_{\mathrm{id}} \cap \Delta^{\mathfrak{p}}_*(\mathrm{rec}^{\mathfrak{p},\ell,\infty}_{H/F,\mathfrak{f}} \cap \vartheta^{\mathfrak{p}}_{b_i}))).$$

Applying the calculation in (45) we can then observe

$$u_2 = (-1)^{n-1} \sum_{i=1}^s c_{\mathrm{id}} \cap ((\omega_{\mathfrak{f},\mathfrak{b}_i,\lambda} \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i)) \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})})$$

$$= (-1)^{n-1} (c_{\mathrm{id}} \cap ((\sum_{i=1}^s \omega_{\mathfrak{f},\mathfrak{b}_i,\lambda} \otimes \mathrm{rec}_{H/F}(\mathfrak{b}_i)) \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})}))$$

$$= (-1)^{n-1} (c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\lambda} \cap \eta_{\mathfrak{p},E_+(\mathfrak{f})}))$$

$$= u_3.$$

Here, the third equality is due to the definition of $\omega_{\mathfrak{f},\lambda}$ from (15). The final equality is simply the definition of u_3 and the fact that $(-1)^{n-1} = (-1)^{n+1}$. $\qquad \square$

8 | EQUALITY OF u_1 AND u_3

In this section, we prove the following theorem.

Theorem 8.1. *We have $u_1 = u_3$.*

We show that for each $\sigma \in G$ we have $u_1(\sigma) = u_3(\sigma)$. This is done by using a strong enough compatibility property which forces the formulas to be equal. A special argument will be required in the case that R contains no finite places, that is, $R = R_\infty$.

We are given a CM abelian extension H/F of conductor $\mathfrak{f}$ such that $\mathfrak{p}$ splits completely in H. Let $\mathfrak{f}'$ be an auxiliary ideal of $\mathcal{O}_F$ that is divisible only by primes dividing $\mathfrak{f}$. Let H' be another finite abelian CM extension of F in which $\mathfrak{p}$ splits completely, such that the conductor of H'/F divides $\mathfrak{f}\mathfrak{f}'$. In particular, the extension H'/F is unramified outside R.

Let $\sigma \in G$. Write $u_1(\sigma, H)$ and $u_3(\sigma, H)$ for σ components of the formulas u_1 and u_3, for the extension H/F and Galois group element σ. We show that, for $i = 1, 3$,

$$u_i(\sigma, H) = \prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_i(\tau, H'). \tag{46}$$

We refer to (46) as norm compatibility.

Proposition 8.2. *We have*

$$u_1(\sigma, H) \equiv u_3(\sigma, H) \quad (\mathrm{mod}\ E_+(\mathfrak{f})).$$

Proof. Let V be a finite index subgroup of $E_+(\mathfrak{f})$ satisfying the conditions given in the statement of Proposition 3.8. Furthermore, we choose V such that if $V = \langle \varepsilon_1, \ldots, \varepsilon_{n-1} \rangle$ then the ε_i along with π satisfy Lemma 4.3. We recall from Section 6.2 the explicit description of $u_3(V, \sigma)$,

$$u_3(V, \sigma) = c_{\mathrm{id}} \cap (\omega_{\mathfrak{f}, \mathfrak{b}, \lambda, V}^{\mathfrak{p}} \cap \eta_{\mathfrak{p}, V}) = \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b}, B_i, \pi\mathcal{O}_{\mathfrak{p}}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, B, \mathcal{O}_{\mathfrak{p}}, 0)} \fint_{\mathbb{O}} x\, d(\zeta_{R,\lambda}(\mathfrak{b}, B, x, 0))(x).$$

We have defined

$$u_1(V, B, \sigma) = \prod_{\varepsilon \in V} \varepsilon^{\zeta_{R,\lambda}(\mathfrak{b}, \varepsilon B \cap \pi^{-1}B, \mathcal{O}_{\mathfrak{p}}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, B, \mathcal{O}_{\mathfrak{p}}, 0)} \fint_{\mathbb{O}} x\, d\nu_\lambda(\mathfrak{b}, B, x),$$

where

$$B = \bigcup_{\tau \in S_{n-1}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \cdots \mid \varepsilon_{\tau(n-1)}]).$$

Thus, $u_1(V, B, \sigma) \equiv u_3(V, \sigma) \pmod{E_+(\mathfrak{f})}$ and hence there exists $\alpha \in F_{\mathfrak{p}}^*$ such that

$$\alpha u_1(\sigma, H) = u_3(\sigma, H) \quad \text{and} \quad \alpha^{[E_+:V]} \in E_+(\mathfrak{f}). \tag{47}$$

Note that here we are also using Proposition 3.7. By Lemma 4.7, we can choose W to be a finite index subgroup of $E_+(\mathfrak{f})$ satisfying the same conditions as V but with $[E_+(\mathfrak{f}) : W]$ coprime to $[E_+(\mathfrak{f}) : V]$. Thus, we also have $\alpha^{[E_+(\mathfrak{f}):W]} \in E_+(\mathfrak{f})$, which combines with (47) to yield $\alpha \in E_+(\mathfrak{f})$ as desired. $\qquad \square$

Assuming that (46) holds we can prove the following.

Proposition 8.3. *Suppose that (46) holds and that $R \neq R_\infty$. Then,*

$$u_1 = u_3.$$

Proof. From Proposition 8.2, we have that for each $\tau \in G'$,

$$u_1(\tau, H') \equiv u_3(\tau, H') \pmod{E_+(\mathfrak{f}\mathfrak{f}')}.$$

Our assumption that (46) holds then gives that for each $\sigma \in G$,

$$u_1(\sigma, H) \equiv u_3(\sigma, H) \pmod{E_+(\mathfrak{f}\mathfrak{f}')}.$$

Since $R \neq R_\infty$, we have

$$\bigcap_{\mathfrak{f}'} E_+(\mathfrak{f}\mathfrak{f}') = \{1\}.$$

Here, the intersection is taken over all possible ideals $\mathfrak{f}'$ divisible only by primes dividing $\mathfrak{f}$. Thus, we have

$$u_1(\sigma, H) = u_3(\sigma, H). \qquad \square$$

Remark 8.4. If $R = R_\infty$ then $\mathfrak{f} = \mathfrak{f}\mathfrak{f}' = 1$ for all possible extensions. Hence, the proof of Proposition 8.3 does not apply.

To handle the case $R = R_\infty$, we extend the definition of u_1 to work with the trivial extension. For a Shintani set $\mathcal{D}$ and compact open $U \subseteq \mathcal{O}_\mathfrak{p}$, we define

$$\nu_\lambda(\mathcal{D}, U) = \zeta_{R,\lambda}(\mathcal{O}_F, \mathcal{D}, U, 0).$$

It is clear that

$$\nu_\lambda(\mathcal{D}, U) = \sum_{\sigma_\mathfrak{b} \in G} \nu_\lambda(\mathfrak{b}, \mathcal{D}, U). \tag{48}$$

We then define, for a Shintani domain $\mathcal{D}$,

$$u_1(F) = \left(\prod_{\epsilon \in E_+} \epsilon^{\nu_\lambda(\epsilon \mathcal{D} \cap \pi^{-1} \mathcal{D}, \mathcal{O}_\mathfrak{p})} \right) \pi^{\nu_\lambda(\mathcal{D}, \mathcal{O}_\mathfrak{p})} \fint_{\mathbb{O}} x \, d\nu_\lambda(\mathcal{D}, x). \tag{49}$$

By (48) and since $\mathfrak{f} = 1$ we have

$$u_1(F) = \prod_{\sigma \in G} u_1(\sigma, H).$$

Lemma 8.5. *We have*

$$u_1(F) = 1.$$

Proof. Since $\mathcal{D}$ is a Shintani domain we have

$$\nu_\lambda(\mathcal{D}, \mathcal{O}_\mathfrak{p}) = \zeta_{R,\lambda}(F/F, \mathcal{O}_F, 0) = 0. \tag{50}$$

Therefore, the π-power term in (49) vanishes. Next, we write

$$\fint_{\mathbb{O}} x\, d\nu_\lambda(D, x) = \frac{\fint_{\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(D, x)}{\fint_{\pi\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(D, x)}. \tag{51}$$

By Lemma 2.10, we calculate

$$\fint_{\pi\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(D, x) = \pi^{\nu_\lambda(\pi D, \pi\mathcal{O}_\mathfrak{p})} \fint_{\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(\pi^{-1}D, x) = \fint_{\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(\pi^{-1}D, x) \tag{52}$$

since $\nu_\lambda(\pi D, \pi\mathcal{O}_\mathfrak{p}) = 0$ as in (50). Since D is a Shintani domain we can write

$$\pi^{-1}D = \bigcup_{\epsilon \in E_+} (\epsilon D \cap \pi^{-1}D).$$

We then have

$$\fint_{\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(\pi^{-1}D, x) = \prod_{\epsilon \in E_+} \left(\fint_{\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(\epsilon D \cap \pi^{-1}D, x) \right)$$

$$= \left(\prod_{\epsilon \in E_+} \epsilon^{\nu_\lambda(\epsilon D \cap \pi^{-1}D, \mathcal{O}_\mathfrak{p})} \right) \fint_{\mathcal{O}_\mathfrak{p}} x\, d\nu_\lambda(D, x). \tag{53}$$

Combining (51), (52), and (53) yields

$$\fint_{\mathbb{O}} x\, d\nu_\lambda(D, x) = \left(\prod_{\epsilon \in E_+} \epsilon^{\nu_\lambda(\epsilon D \cap \pi^{-1}D, \mathcal{O}_\mathfrak{p})} \right)^{-1}$$

Applying the definition of $u_1(F)$ yields the desired result. $\qquad\square$

Proposition 8.6. *Suppose that (46) holds and that $R = R_\infty$. Then,*

$$u_1 = u_3.$$

Proof. Let $\sigma \in G$. By Proposition 8.2 there exists $\varepsilon(\sigma) \in E_+$ such that

$$u_1(\sigma) = \varepsilon(\sigma) u_3(\sigma).$$

Let $\mathfrak{r}$ be a prime of F. From the equation

$$\zeta_{R \cup \{\mathfrak{r}\}}(\mathfrak{b}, D, U, s) = \zeta_R(\mathfrak{b}, D, U, s) - N\mathfrak{r}^{-s} \zeta_R(\mathfrak{b}\mathfrak{r}^{-1}, D, U, s),$$

it follows that

$$u_1(S \cup \{\mathfrak{r}\}, \sigma) = u_1(S, \sigma) u_1(S, \sigma_\mathfrak{r}^{-1}\sigma)^{-1}.$$

Proposition 5.2 *c)* gives the same result for u_2. Applying Theorem 7.1 we have the same result for u_3, therefore

$$u_3(S, \sigma) u_3(S, \sigma_\mathfrak{r}^{-1}\sigma)^{-1} = u_3(S \cup \{\mathfrak{r}\}, \sigma)$$

$$= u_1(S \cup \{\mathfrak{r}\}, \sigma)$$

$$= u_1(S, \sigma) u_1(S, \sigma_{\mathfrak{r}}^{-1} \sigma)^{-1}$$

$$= \varepsilon(\sigma) \varepsilon(\sigma \sigma_{\mathfrak{r}}^{-1})^{-1} u_3(S, \sigma) u_3(S, \sigma_{\mathfrak{r}}^{-1} \sigma)^{-1}. \tag{54}$$

Here, (54) is given by Proposition 8.3, which can be applied since we have added $\mathfrak{r}$ to the set R. It follows that $\varepsilon(\sigma) = \varepsilon(\sigma \sigma_{\mathfrak{r}}^{-1})$. Repeating this for all such $\mathfrak{r}$ we see that $\varepsilon(\sigma)$ is independent of $\sigma \in G$. Write $\varepsilon = \varepsilon(\sigma)$. Then

$$1 = u_1(F) = \prod_{\sigma \in G} u_1(\sigma, H) = \varepsilon^{|G|} \prod_{\sigma \in G} u_3(\sigma, H) = \varepsilon^{|G|}.$$

The last equality follows since $\prod_{\sigma \in G} u_2(\sigma, H) = 1$ by 5.2 b),d) and Theorem 7.1. Since $\varepsilon \in E_+$, it follows that $\varepsilon = 1$. This gives the desired result. $\qquad\square$

Theorem 8.1, under the assumption that (46) holds, then follows from the combination of Propositions 8.3 and 8.6. In the next section, we prove the norm compatibility property (46) for u_1 and u_2.

9 | NORM COMPATIBILITY RELATIONS

In this section, we prove norm compatibility properties for u_1 and u_3.

9.1 | Norm compatibility for u_1

In this section, we again allow any choice of appropriate T.

The reciprocity map identifies $\mathrm{Gal}(H'/H)$ with

$$\{\beta \in (\mathcal{O}_F/\mathfrak{f}\mathfrak{f}')^* \mid \beta \equiv 1 \pmod{\mathfrak{f}}\}/E_+(\mathfrak{f})_{\mathfrak{p}}. \tag{55}$$

We let $\mathcal{D}_{\mathfrak{f}}$ be a Shintani domain for $E_+(\mathfrak{f})$ and define

$$\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma \mathcal{D}_{\mathfrak{f}},$$

where the union is over a set of representatives $\{\gamma\}$ for $E_+(\mathfrak{f}\mathfrak{f}')$ in $E_+(\mathfrak{f})$. Let e' be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}\mathfrak{f}'}$, and suppose that $\mathfrak{p}^{e'} = (\pi')$ with π' totally positive and $\pi' \equiv 1 \pmod{\mathfrak{f}\mathfrak{f}'}$. We can choose π' such that $\pi' = \pi^\alpha$ for some $\alpha \geqslant 1$. We then define $\mathcal{O}' = \mathcal{O}_{\mathfrak{p}} - \pi' \mathcal{O}_{\mathfrak{p}}$.

Let B denote a set of totally positive elements of $\mathcal{O}_F$ that are relatively prime to S and $\overline{T}$ and whose images in $(\mathcal{O}_F/\mathfrak{f}\mathfrak{f}')^*$ are a set of distinct representatives for (55).

The following theorem is stated without proof by the first author in [4, Theorem 7.1]. For completeness we include a proof of this result here.

Theorem 9.1 Theorem 7.1, [4]. *We have*

$$u_T(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}) = \prod_{\beta \in B} u_T(\mathfrak{b}(\beta), \beta^{-1} \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}).$$

The key to the proof of Theorem 9.1 is to use translation properties of Shintani sets. For a subset A of equivalence classes of (55), let $\nu_T^A(\mathfrak{b}, D, U) = \zeta_{R,T}^A(\mathfrak{b}, D, U, 0)$, where ζ_R^A is the zeta function

$$\zeta_R^A(\mathfrak{b}, D, U, s) = N\mathfrak{b}^{-s} \sum_{\substack{\alpha \in \mathfrak{b}^{-1} \cap D, \ \alpha \in U \\ \alpha \in A, \ (\alpha, R) = 1}} N\alpha^{-s}.$$

This definition extends to $\zeta_{R,T}^A$ as in (7). Throughout this section, we will use the following simple equality:

$$\nu_T^{\{\pi^{-1}\}}(\mathfrak{b}, D, U) = \nu_T^{\{1\}}(\mathfrak{b}, \pi D, \pi U).$$

This follows from Lemma 2.10. Recall the following definition. For $\beta \in B$,

$$u_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}) = \epsilon(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}, \pi')(\pi')^{\zeta_{R,T}(H_{\mathfrak{ff'}}/F, \mathfrak{b}(\beta), 0)} \fint_{\mathbb{O}'} x \, d\nu_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}, x).$$

It is clear from the definition of B that Theorem 9.1 follows from the following proposition.

Proposition 9.2. *Let $\beta \in B$. We have*

$$u_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}) = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T^B(\mathfrak{b}(\beta), \epsilon \beta^{-1} D_{\mathfrak{f}} \cap \pi^{-1} \beta^{-1} D_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \right) \pi^{\nu_T^B(\mathfrak{b}(\beta), D_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \fint_{\mathbb{O}} x \, d\nu_T^B(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{f}}, x).$$

The proof of Theorem 9.2 is largely an exercise in explicit calculation. We begin by considering the multiplicative integral in $u_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}})$.

Lemma 9.3. *We have*

$$\fint_{\mathbb{O}'} x \, d\nu_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}, x) = \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), D_{\mathfrak{ff'}}, \pi^i \mathbb{O})} \right) \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{ff'})} \epsilon^{\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon \beta^{-1} D_{\mathfrak{ff'}} \cap \pi^{-i} \beta^{-1} D_{\mathfrak{ff'}}, \mathbb{O})} \right)$$

$$\left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{ff'})} \gamma^{\nu_T^A(\mathfrak{b}(\beta), \gamma \beta^{-1} D_{\mathfrak{f}}, \mathbb{O})} \right) \fint_{\mathbb{O}} x \, d\nu_T^B(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{f}}, x).$$

Proof. Since $\pi' = \pi^\alpha$ and $\mathbb{O}' = \mathcal{O}_{\mathfrak{p}} - \pi' \mathcal{O}_{\mathfrak{p}}$ we have $\mathbb{O}' = \bigcup_{i=0}^{\alpha-1} \pi^i \mathbb{O}$. Then

$$\fint_{\mathbb{O}'} x \, d\nu_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}, x) = \prod_{i=0}^{\alpha-1} \fint_{\pi^i \mathbb{O}} x \, d\nu_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}, x).$$

By changing variables and then factoring out π^i we have

$$I(\beta) := \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), D_{\mathfrak{ff'}}, \pi^i \mathbb{O})} \right) \prod_{i=0}^{\alpha-1} \fint_{\mathbb{O}} x \, d\nu_T(\mathfrak{b}(\beta), \beta^{-1} D_{\mathfrak{ff'}}, \pi^i x)$$

$$= \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), D_{\mathfrak{ff'}}, \pi^i \mathbb{O})} \right) \prod_{i=0}^{\alpha-1} \fint_{\mathbb{O}} x \, d\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \pi^{-i} \beta^{-1} D_{\mathfrak{ff'}}, x).$$

We now note that we can write, for $i = 1, \ldots, \alpha - 1$,

$$\pi^{-i} \mathcal{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} (\epsilon \mathcal{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i} \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}).$$

Then,

$$\prod_{i=0}^{\alpha-1} \oint_{\mathbb{O}} x \, d\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x)$$

$$= \prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \oint_{\mathbb{O}} x \, d\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x)$$

$$= \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})} \right) \oint_{\mathbb{O}} x \, d\nu_T^A(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x).$$

Here, $A = \{1, \pi^{-1}, \ldots, \pi^{\alpha-1}\}$. Then since $\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma \mathcal{D}_{\mathfrak{f}}$ we can write

$$\oint_{\mathbb{O}} x \, d\nu_T^A(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, x) = \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_T^A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})} \right) \oint_{\mathbb{O}} x \, d\nu_T^{\langle A,E \rangle}(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}}, x)$$

where $E = E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')$. Thus, we have, noting that $B = AE = \{ae \mid a \in A, e \in E\}$,

$$I(\beta) = \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})} \right) \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})} \right)$$

$$\left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_T^A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})} \right) \oint_{\mathbb{O}} x \, d\nu_T^B(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}}, x). \qquad \square$$

We now consider the powers of π given in the definition of $u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'})$ and arising in the statement of Lemma 9.3. Recall that $\pi' = \pi^\alpha$.

Lemma 9.4.

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})} \right) \pi^{\alpha\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)} = \pi^{\nu_T^B(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \mathbb{O}_{\mathfrak{p}})}.$$

Proof. Since $\pi^i\mathbb{O} = \pi^i\mathcal{O}_{\mathfrak{p}} - \pi^{i+1}\mathcal{O}_{\mathfrak{p}}$ we have by a telescope argument

$$\sum_{i=1}^{\alpha-1} i\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O}) = -(\alpha-1)\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^\alpha\mathcal{O}_{\mathfrak{p}}) + \sum_{i=1}^{\alpha-1} \nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathcal{O}_{\mathfrak{p}}).$$

Recalling the definition of $\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}$ we also note that for $i = 0, \ldots, \alpha - 1$ we have

$$\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathcal{O}_{\mathfrak{p}}) = \nu_T^E(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \pi^i\mathcal{O}_{\mathfrak{p}}).$$

Thus, we can calculate, using the fact that $\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0) = \nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \mathcal{O}_{\mathfrak{p}})$,

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})}\right) \pi^{\alpha\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)}$$

$$= \left(\prod_{i=1}^{\alpha-1} \pi^{\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathcal{O}_{\mathfrak{p}})}\right) \pi^{-(\alpha-1)\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^\alpha\mathcal{O}_{\mathfrak{p}})} \pi^{\alpha\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \mathcal{O}_{\mathfrak{p}})}$$

$$= \left(\prod_{i=1}^{\alpha-1} \pi^{\nu_T^E(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \pi^i\mathcal{O}_{\mathfrak{p}})}\right) \pi^{-(\alpha-1)\nu_T^E(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \pi^\alpha\mathcal{O}_{\mathfrak{p}})} \pi^{\alpha\nu_T^E(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})}.$$

By Lemma 2.10, we have for $i = 1, \ldots, \alpha$,

$$\nu_T^E(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \pi^i\mathcal{O}_{\mathfrak{p}}) = \nu_T^{E, \{\pi^{-i}\}}(\mathfrak{b}(\beta), \pi^{-i}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}}).$$

We can then write $\pi^{-i}\mathcal{D}_{\mathfrak{f}} = \bigcup_{\delta \in E_+(\mathfrak{f})} \delta\mathcal{D}_{\mathfrak{f}} \cap \pi^{-i}\mathcal{D}_{\mathfrak{f}}$. Then

$$\nu_T^{E, \{\pi^{-i}\}}(\mathfrak{b}, \pi^{-i}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}}) = \sum_{\delta \in E_+(\mathfrak{f})} \nu_T^{E, \{\pi^{-i}\}}(\mathfrak{b}, \delta\mathcal{D}_{\mathfrak{f}} \cap \pi^{-i}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})$$

$$= \sum_{\delta \in E_+(\mathfrak{f})} \nu_T^{E, \{\pi^{-i}\}}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}} \cap \delta\pi^{-i}\mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})$$

$$= \nu_T^{E, \{\pi^{-i}\}}(\mathfrak{b}, \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}}).$$

Remarking that $\pi^{-\alpha} = (\pi')^{-1} \equiv 1 \pmod{\mathfrak{f}\mathfrak{f}'}$, we deduce that

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu_T(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i\mathbb{O})}\right) \pi^{\alpha\zeta_{R,T}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)} = \prod_{i=0}^{\alpha-1} \pi^{\nu_T^{E, \{\pi^{-i}\}}(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} = \pi^{\nu_T^{\langle A, E\rangle}(\mathfrak{b}(\beta), \mathcal{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})}.$$

Noting that $B = AE$ completes the proof. $\qquad\square$

We now consider the error term in the definition of $u_T(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'})$ and the products of elements of $E_+(\mathfrak{f})$ that arise in Lemma 9.3. Considering Lemmas 9.3 and 9.4, in order to prove Proposition 9.2 it is enough to prove the following.

Proposition 9.5. *Let*

$$\mathrm{Err}(\beta) = \epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \pi') \left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathcal{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})}\right)$$

$$\times \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_T^A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathcal{D}_{\mathfrak{f}}, \mathbb{O})}\right).$$

Then

$$\mathrm{Err}(\beta) = \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T^B(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-1}\beta^{-1}D_\mathfrak{f}, \mathscr{O}_\mathfrak{p})}.$$

For clarity, we shall perform the calculations required for this proposition in a few lemmas.

Lemma 9.6. *We have*

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T^E(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}D_\mathfrak{f}, \mathscr{O}_\mathfrak{p})} \right) \left(\prod_{i=1}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-i}\beta^{-1}D_\mathfrak{f}, \mathbb{O})} \right).$$

Proof. Considering the definition of $D_{\mathfrak{f}\mathfrak{f}'}$ we calculate

$$\epsilon(\mathfrak{b}(\beta), \beta^{-1}D_{\mathfrak{f}\mathfrak{f}'}, \pi') \tag{56}$$

$$= \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_T(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_\mathfrak{p})} \tag{57}$$

$$= \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_T(\mathfrak{b}(\beta), \epsilon\gamma\beta^{-1}D_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_\mathfrak{p})} \tag{58}$$

$$= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu_T(\mathfrak{b}(\beta), \gamma\beta^{-1}D_\mathfrak{f}, \mathscr{O}_\mathfrak{p})} \right) \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_\mathfrak{p})} \right). \tag{59}$$

Similarly we have

$$\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_\mathfrak{p})}$$

$$= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_T(\mathfrak{b}(\beta), \gamma\pi^{-\alpha}\beta^{-1}D_\mathfrak{f}, \mathscr{O}_\mathfrak{p})} \right) \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T^E(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}D_\mathfrak{f}, \mathscr{O}_\mathfrak{p})} \right). \tag{60}$$

We also calculate for $i = 1, \dots, \alpha - 1$

$$\prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}D_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})}$$

$$= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\beta^{-1}D_\mathfrak{f}, \mathbb{O}) + \nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\pi^{-i}\beta^{-1}D_\mathfrak{f}, \mathbb{O})} \right)$$

$$\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}D_\mathfrak{f} \cap \pi^{-i}\beta^{-1}D_\mathfrak{f}, \mathbb{O})}. \tag{61}$$

We now note the following equalities, both of which hold via telescoping sum arguments.

1.

$$\prod_{i=1}^{\alpha-1}\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{-\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\gamma\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}\right)\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_T^A(\mathfrak{b}(\beta),\gamma\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}\right)$$

$$=\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_T(\mathfrak{b}(\beta),\gamma\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}$$

2.

$$\prod_{i=1}^{\alpha-1}\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_T^{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\gamma\pi^{-i}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}\right)$$

$$=\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_T(\mathfrak{b}(\beta),\gamma\beta^{-1}D_{\mathfrak{f}},\pi\mathscr{O}_{\mathfrak{p}})-\nu_T^{\{\pi^{-\alpha}\}}(\mathfrak{b}(\beta),\gamma\pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathscr{O}_{\mathfrak{p}})}$$

$$=\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_T(\mathfrak{b}(\beta),\gamma\beta^{-1}D_{\mathfrak{f}},\pi\mathscr{O}_{\mathfrak{p}})-\nu_T(\mathfrak{b}(\beta),\gamma\pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathscr{O}_{\mathfrak{p}})}.$$

Combining these two equalities with the calculations in (59), (60), and (61) gives the result. $\qquad\square$

If $\alpha = 1$, then Lemma 9.6 is equivalent to Proposition 9.5 and thus we are finished in the case $\alpha = 1$. From this point on, we assume that $\alpha > 1$.

Lemma 9.7. *If $\alpha > 1$ then*

$$\mathrm{Err}(\beta)=\left(\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{\nu_T^B(\mathfrak{b}(\beta),\epsilon\beta^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}},\mathscr{O}_{\mathfrak{p}})}\right)\left(\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathscr{O}_{\mathfrak{p}})}\right)$$

$$\prod_{i=1}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{-\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}},\pi\mathscr{O}_{\mathfrak{p}})}$$

$$\prod_{i=2}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}.$$

Proof. For $i = 2, \ldots, \alpha$ we have

$$\pi^i D_{\mathfrak{f}} = \bigcup_{\delta\in E_+(\mathfrak{f})}\pi^{-1}\delta D_{\mathfrak{f}}\cap\pi^{-i}D_{\mathfrak{f}}.$$

Thus, applying this to the result of Lemma 9.6, we have

$$\mathrm{Err}(\beta)=\left(\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{\nu_T^E(\mathfrak{b}(\beta),\epsilon\beta^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}},\mathscr{O}_{\mathfrak{p}})}\right)\left(\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathscr{O}_{\mathfrak{p}})}\right)$$

$$\prod_{i=1}^{\alpha-1}\left(\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}\right.$$

$$\left.\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}\right).$$

Remarking that $\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^{\langle E,\{\pi^{-1}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}=1$, since $\epsilon\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}}=\emptyset$, gives the result. $\qquad\square$

If $\alpha=2$, it is straightforward to see that Lemma 9.7 is equivalent to Proposition 9.5 and thus we are also finished in the case $\alpha=2$. From this point on, we assume that $\alpha>2$. From Lemma 9.7, one can see that to prove Proposition 9.5 it is enough for us to show

$$1=\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}$$

$$\prod_{i=1}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{-\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}D_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}D_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})}$$

$$\prod_{i=2}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}D_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}. \tag{62}$$

To do this, we first show the following lemma.

Lemma 9.8. *We have that for $j=1,\dots,\alpha-1$ the right-hand side of (62) is equal to*

$$e(j)=\left(\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}D_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}\right)$$

$$\prod_{i=j}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{-\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(j-1)}D_{\mathfrak{f}}\cap\pi^{-j}\beta^{-1}D_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})}$$

$$\prod_{i=j+1}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_T^{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-j}D_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}D_{\mathfrak{f}},\mathbb{O})}.$$

Note that for $j=\alpha-1$ the last product is empty. We also remark that it is implicit in the statement of this lemma that $e(1)=\dots=e(\alpha-1)$.

Proof. We prove this by induction. The case $j=1$ holds trivially. We now assume it holds for j and prove the result for $j+1$, that is, we show $e(j)=e(j+1)$. To do this we note that for $i=j+2,\dots,\alpha$, we have

$$\pi^{-i}D_{\mathfrak{f}}=\bigcup_{\kappa\in E_+(\mathfrak{f})}\pi^{-(j+1)}\kappa D_{\mathfrak{f}}\cap\pi^{-i}D_{\mathfrak{f}}.$$

Thus, $e(j)$ is equal to the product of the following elements:

$$\left(\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}D_{\mathfrak{f}} \cap \pi^{-(j+1)}\beta^{-1}D_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \right) \left(\prod_{\kappa \in E_+(\mathfrak{f})} \kappa^{\nu_T^E(\mathfrak{b}(\beta), \kappa\beta^{-1}\pi^{-(j+1)}D_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \right) \tag{63}$$

$$\prod_{i=j}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-(j-1)}D_{\mathfrak{f}} \cap \pi^{-j}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})} \tag{64}$$

$$\prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}D_{\mathfrak{f}} \cap \pi^{-(j+1)}\beta^{-1}D_{\mathfrak{f}}, \mathbb{O})} \tag{65}$$

$$\prod_{i=j+2}^{\alpha-1} \prod_{\kappa \in E_+(\mathfrak{f})} \kappa^{\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \kappa\beta^{-1}\pi^{-(j+1)}D_{\mathfrak{f}} \cap \pi^{-i}\beta^{-1}D_{\mathfrak{f}}, \mathbb{O})}. \tag{66}$$

We remark that the first bracketed term in (63), and (66) are already products in $e(j+1)$. We now consider (65) and calculate that it is equal to

$$\left(\prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^{\langle E, \{\pi^{-(i-1)}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-(j-1)}D_{\mathfrak{f}} \cap \pi^{-j}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})} \right)$$

$$\left(\prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{-\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}D_{\mathfrak{f}} \cap \pi^{-(j+1)}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})} \right). \tag{67}$$

We now consider the way the terms in (67) interact with (64). Multiplying (67) by (64) gives

$$\left(\prod_{i=j+1}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_T^{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-j}D_{\mathfrak{f}} \cap \pi^{-(j+1)}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})} \right)$$

$$\left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_T^{\langle E, \{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-(j-1)}D_{\mathfrak{f}} \cap \pi^{-j}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})} \right). \tag{68}$$

The first term in (68) is the term we were missing from $e(j+1)$. Thus, it only remains to show that the second bracketed term in (63) multiplied by the second bracketed term in (68) is equal to 1. This is shown by the following calculation,

$$\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}D_{\mathfrak{f}} \cap \pi^{-(j+1)}\beta^{-1}D_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})}$$

$$= \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^{\langle E, \{\pi\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-(j-1)}D_{\mathfrak{f}} \cap \pi^{-j}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})}$$

$$= \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^{\langle E, \{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-(j-1)}D_{\mathfrak{f}} \cap \pi^{-j}\beta^{-1}D_{\mathfrak{f}}, \pi\mathcal{O}_{\mathfrak{p}})}.$$

We therefore deduce that

$$e(j) = e(j + 1)$$

as claimed. This completes the proof of the lemma.   □

We are now ready to prove Proposition 9.5.

Proof of Proposition 9.5. We consider $e(\alpha - 1)$. From Lemma 9.8, we have that $e(\alpha - 1)$ is equal to the right hand side of (62). Then

$$e(\alpha - 1) = \left(\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_T^E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-(\alpha-1)}D_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}D_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})} \right)$$

$$\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_T^{\langle E,\{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(\alpha-2)}D_{\mathfrak{f}} \cap \pi^{-(\alpha-1)}\beta^{-1}D_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})}.$$

Since $\pi \equiv \pi^{-(\alpha-1)} \pmod{\mathfrak{f}\mathfrak{f}'}$, it is clear that

$$e(\alpha - 1) = 1.$$

This completes the proof of Proposition 9.5 and thus proves Theorem 9.1.   □

9.2 │ Norm compatibility for u_3

We prove norm compatibility for u_2 and then obtain the result for u_3 by Theorem 7.1.
  We recall the definition

$$u_2 = \sum_{\sigma \in G} u_2(\sigma) \otimes [\sigma^{-1}] = \mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H/F}).$$

Theorem 9.9. *We have for any $\sigma \in G$,*

$$u_2(\sigma, H) = \prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_2(\tau, H').$$

Remark 9.10. This theorem has been stated without proof by the first and third authors in [8, Proposition 6.3]. We include the proof for completeness. We note also that the proof of the norm compatibility for u_2 is much simpler than that for u_1. This is a result of the additional structure we have due to the cohomological nature of the construction.

Proof of Theorem 9.9. We consider the natural map

$$\psi : F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G'] \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$$

$$\sum_{\tau \in G'} n_\tau \otimes [\tau] \mapsto \sum_{\sigma \in G} \left(\prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} n_\tau \right) \otimes [\sigma].$$

Then, on one hand,

$$\psi(u_2(H')) = \sum_{\sigma \in G} \Big(\prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_2(\tau, H') \Big) \otimes [\sigma].$$

On the other hand,

$$\psi(u_2(H')) = \psi(\mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H'/F}))$$
$$= \mathrm{Eis}_F^0 \cap \psi_* \Delta_*(c_{\mathrm{id}} \cap \rho_{H'/F})$$
$$= \mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}} \cap \psi_* \rho_{H'/F}).$$

The only equality of note here is the final one. This follows since we can commute ψ_* with Δ_*, which is a consequence of the definitions in Section 5.1. Then since $\psi_* \rho_{H'/F} = \rho_{H/F}$, the desired result follows. $\qquad\square$

ORCID

Matthew H. L. Honnor https://orcid.org/0000-0002-8041-665X

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